Week 2: Rayleigh-Taylor instability of a suspended thin viscous liquid film

We consider a thin (nominal height h_0) liquid film underneath a horizontal rigid plate. The film has dynamic viscosity μ, density ρ and surface tension with air γ. We aim at predicting the most unstable wavelenght as this film destabilizes under the action of gravity. The air is considered at rest and constant pressure p_0, which implies that the effect of the density on the base pressure gradient is neglected. There is therefore no Archimedes force. The statement that the liquid film is very thin should first be made precise: we consider $h_0 \ll \ell_c$, where $\ell_c = \sqrt{\rho g/\gamma}$ is the so called capillary length. Without justifying it fully, we will also neglect inertial effects. We focus first on the two-dimensional idealization of the flow. We introduce a reference frame with y pointing down away from the plate located in $y = 0$ and x the horizontal axis parallel to the plane. The velocity vector is $\mathbf{u} = (u; v)$. The free surface of the film is denoted by $h(x, t)$.

- Derivation of the lubrication equation governing the thin film thickness distribution (called the Reynolds equation)

  1. We scale x with ℓ_c, y with h_0, which naturally yields two different scales for uOne can show that the Navier-Stokes equations reduce in this limit to the following system

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{1}$$

$$\frac{\partial p}{\partial x} - \mu \frac{\partial^2 u}{\partial y^2} = 0, \tag{2}$$

$$\frac{\partial p}{\partial y} - \rho g = 0, \tag{3}$$

  First show that

$$p(x, y) = p(h) - \rho g(h - y) \tag{4}$$

  and therefore that $\frac{\partial p}{\partial x}$ does not depend on y!

  2. The expression of the stress tensor in this lubrication limit is

$$\boldsymbol{\sigma} = \begin{pmatrix} -p & \mu\frac{\partial u}{\partial y} \\ \mu\frac{\partial u}{\partial y} & -p \end{pmatrix}. \tag{5}$$

  What is the boundary condition for u valid on the film surface?

  Using this boundary condition on the film interface together with the boundary condition pertaining on the wall, integrate these equations to show that

$$u(x, y) = \frac{1}{2\mu}\frac{\partial p}{\partial x} y(y - 2h). \tag{6}$$

  3. Show that

$$v(x, h) = \frac{1}{3\mu}\frac{\partial^2 p}{\partial x^2} h^3 + \frac{1}{2\mu}\frac{\partial p}{\partial x}\frac{\partial h}{\partial x} h^2. \tag{7}$$

4. Use the kinematic equation

$$\frac{\partial h}{\partial t} = v(x,h) - u(x,h)\frac{\partial h}{\partial x} \tag{8}$$

to demonstrate that

$$\frac{\partial h}{\partial t} = \frac{1}{3\mu}\frac{\partial}{\partial x}(h^3\frac{\partial p}{\partial x}). \tag{9}$$

5. Denoting by κ the interface curvature, write the boundary condition for $p(y = h)$, assuming that viscous normal stresses can be neglected, as given by the expression of the stress tensor.
   Deduce $\frac{\partial p}{\partial x}$.

- Derivation of the dispersion relation

1. We introduce the following normal mode ansatz $h = h_0 + \epsilon \hat{h} \exp(i(kx - \omega t))$. What is the expression of the linearized curvature?
2. Demonstrate that

$$\omega = i\frac{\rho g h_0^3}{3\mu}(k^2 - \ell_c^2 k^4). \tag{10}$$

Plot the growth-rate and determine the cut-off wavenumber as well as the wavelength of maximum growth.