Instability of a thin suspended film

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We consider a thin liquid film of thickness h, dynamic viscosity μ and density ρ situated on the underside of a horizontal plane in a gravitational field $\mathbf{g} = -g\mathbf{e}_y$. We study the stability of this film with respect to bidimensional perturbations in the vertical plane (x, y). The unperturbed free surface of the film is defined by $y = 0$ (the solid surface is therefore at $y = h$), the pressure being therefore $\overline{P} = P_0 - \rho g y$ where P_0 is the air pressure below the film.

We consider perturbations having a wavelength large with respect to the film thickness : $kh \ll 1$ ($\partial_x u + \partial_y v = 0$ implies $v \sim (kh)u \ll u$, the flow is quasi-parallel). Furthermore, we consider the flow quasi-static, *i.e.* it adapts very quickly to the boundary conditions : time only enters through the deformed free surface condition $\eta(x,t) = \hat{\eta}e^{i(kx-\omega t)} + c.c.$. The linearized equations of the perturbations of the base state read therefore

$$0 = -\partial_x p + \mu \partial_{yy} u \,, \quad 0 = -\partial_y p \,. \tag{1}$$

1. We remind that the components of the stress tensor of a newtonian fluid are given by $\sigma_{ij} = -P\delta_{ij} + \mu(\partial_{x_j} U_i + \partial_{x_i} U_j)$. Show that, if considering the air below the film only imposing its pressure P_0 (free surface condition), the interface conditions of the tangential and normal stresses across the free surface $y = \eta(x,t)$, once linearized around $y = 0$, read

$$\partial_y u = 0 \,, \quad p - \rho g \eta = \gamma \partial_{xx} \eta \,. \tag{2}$$

2. Using the boundary conditions at the solid surface and the free surface, determine the velocity u as a function of the pressure gradient $\partial_x p$.
3. Determine the velocity v as a function of the pressure gradient $\partial_x p$.
4. Deduce, using the kinematic condition at the interface, the following expression for the growth rate of the perturbations

$$\omega_i = \frac{1}{3\tau} k^2 h^2 (1 - k^2 l_c^2) \,, \quad \tau = \frac{\mu}{\rho g h} \,, \quad l_c = \sqrt{\frac{\gamma}{\rho g}} \,. \tag{3}$$

Comment the result.