Cavitation & Interface Phenomena

Autumn semester 2022

Exercises – Series 3

Dynamics of a spherical bubble

2.1. Establish the Rayleigh-Plesset equation for a spherical bubble placed in infinite volume of liquid.

2.2. By neglecting the effects of viscosity and surface tension and assuming that the bubble is filled only with vapor (Rayleigh equation), derive the expression for the interface velocity and compute the collapse time T_i (Rayleigh time).

2.3. By taking the initial radius R_0 and the Rayleigh time T_i as the length and time scales ($r=R/R_0$, $t^*=t/T_i$), write the Rayleigh equation in non-dimensional form.

2.4. Demonstrate that this non-dimensional equation can be written as:

$$\ddot{r} + \frac{K}{r^4} = 0$$

With a constant K (to be determined) and the dots denote derivatives with respect to the non-dimensional time t*.

2.5. Let us consider the collapse of a cavitation bubble with an initial radium R_0=1 cm, in an infinite volume of liquid at the pressure p_{ref}=1 bar. The bubble contains non-condensable gas at the pressure p_{g0}=100 [Pa]. Using *Matlab,* solve numerically the evolution of the radius with the time according to:

1. The Rayleigh equation (compare the collapse time with the analytical solution)
2. The Rayleigh-Plesset equation

hint: use the Ordinary Differential Equation (ODE) solvers in Matlab to solve numerically the differential equation.

2.6. To model more realistically the bubble rebounds and shockwave emissions during the collapse, it is necessary to take into account the liquid compressibility. This is accomplished using the following equation (Keller-Miksis) :

$$\ddot{R} = \frac{(p_g + p_v - p_\infty)(1+\tilde{v}) + R\dot{p}_g / c - (3-\tilde{v})\dot{R}^2 \rho / 2}{(1-\tilde{v})\rho R} \qquad \textit{with} \qquad \tilde{v}(t) \equiv \dot{R}(t)/c$$

Assuming an adiabatic evlution of the pressure of non-condensable gas in the bubble:

$$p_g = p_{g0} \left(\frac{R_{max}}{R} \right)^{3\gamma}$$

Implement this set of equations in your program and compare the solution with previous results.

2.7. Compute the pressure p_g, the temperature T and the interface velocity $\dot{R}$ during the collapse using the different models (Rayleigh, Rayleigh-Plesset, Keller-Miksis).

2.8. Compare the values previously obtained if the pressure of non-condensable gas is changed to p_{g0}=1000 [Pa]. Repeat the operation with p_{g0}=10'000 [Pa].

2.9. By neglecting the effects of viscosity & surface tension and assuming that the bubble is filled only with vapor, plot the pressure distribution in the liquid when $R/R_0=0.2$. Repeat the operation when $R/R_0=0.1$ and $R/R_0=0.05$.

2.10. Demonstrate that at any location in the surrounding liquid, the local velocity exhibits a maximum when $R/R_0=0.25\hat{}1/3\approx 63\%$. Give an expression of this maximum velocity at any location *r*.

2.11. Check the validity of the analytical solution for the collapse of a bubble proposed by *Nikolay A Kudryashovand Dmitry I Sinelshchikov:*

$$u\ddot{u} + \frac{3}{2}\dot{u}^2 + \xi^2 = 0$$

$$\pm t = \sqrt{\frac{2}{3}}\frac{1}{\xi}\left[\frac{1}{u^3} - 1\right]^{1/2} F\left\{\frac{1}{2}, \frac{4}{3}; \frac{3}{2}; 1 - \frac{1}{u^3}\right\}$$

t and *u* stand for non-dimensional time and radius.
How does the computation time required to evaluate the analytical solution compare with the numerical solution?

References

- **Cavitation lecture notes**
- Obreschkow, D., Bruderer, M. & Farhat, M. **Analytical approximations for the collapse of an empty spherical bubble**. Phys. Rev. E 85, 66303 (2012).
- Kudryashov, N. A. & Sinelshchikov, D. I. **Analytical solutions of the Rayleigh equation for empty and gas-filled bubble**. J. Phys. A: Math. Theory 47, 405202 (2014).

SOLUTION

2.1) write the continuity and momentum equations for an infinite volume of liquid with an empty cavity inside:

Continuity:

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 u_r) + \frac{1}{rsin(\theta)}\frac{\partial}{\partial \theta}(u_\theta \sin(\theta)) + \frac{1}{rsin(\theta)}\frac{\partial u_\phi}{\partial \phi} = 0$$

We only have radial motion: $u_\theta = u_\phi = 0$

Symmetrical geometry: $\frac{\partial}{\partial \theta} = \frac{\partial}{\partial \phi} = 0$

$$\rightarrow \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 u_r) = 0 \quad \rightarrow \quad r^2 u_r = 0 \quad \rightarrow u_r = \frac{A}{r^2}$$

We know that $u_r = u_r(r,t) \xrightarrow{@\, r=R} \dot{R} = \frac{A(t)}{R^2} \rightarrow A(t) = R^2\dot{R}$

$$\rightarrow \quad u_r = \frac{R^2\dot{R}}{r^2}$$

Navier-Stokes equations in r direction:

$$\rho\left(\frac{\partial u_r}{\partial t} + u_r\frac{\partial u_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial u_r}{\partial \theta} + \frac{v_\phi}{rsin(\theta)}\frac{\partial u_r}{\partial \phi} - \frac{v_\theta^2 + v_\phi^2}{r}\right)$$
$$= \frac{-\partial p}{\partial r}$$
$$+ \mu\left(\frac{\partial}{\partial r}\left(\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 u_r)\right) + \frac{1}{r^2(\sin\theta)^2}\frac{\partial}{\partial \theta}\left(\sin(\theta)\frac{\partial u_r}{\partial \theta}\right) + \frac{1}{r^2(\sin\theta)^2}\frac{\partial^2 u_r}{\partial \phi^2}\right.$$
$$\left. - \frac{2}{r^2\sin(\theta)}\frac{\partial}{\partial \theta}(v_\theta \sin(\theta)) - \frac{2}{r^2\sin(\theta)}\frac{\partial v_\phi}{\partial \phi}\right) + pg_r$$

Simplifying the above equation:

$$\rho\left(\frac{\partial u_r}{\partial t}+u_r\frac{\partial u_r}{\partial r}\right)=\frac{-\partial p}{\partial r}+\mu\left(\frac{\partial}{\partial r}\left(\frac{1}{r^2}\frac{\partial}{\partial r}(r^2u_r)\right)\right)$$

Inserting the found form of $u_r=\frac{R^2\dot{R}}{r^2}$, we have:

$$\frac{1}{r^2}\frac{\partial}{\partial t}\left(R^2\dot{R}\right)+\frac{\left(R^2\dot{R}\right)^2}{r^2}\frac{\partial}{\partial r}\left(\frac{1}{r^2}\right)=-\frac{1}{\rho}\frac{\partial p}{\partial r} \qquad , \qquad R=R(t)$$

$$2R\dot{R}^2r^{-2}+\ddot{R}R^2r^{-2}-2R^4\dot{R}^2r^{-5}=-\frac{1}{\rho}\frac{\partial p}{\partial r}$$

Here, the integral of $\int_R^\infty(\dots)dr$ is taken, which simplifies to:

$$\Longrightarrow \qquad R\ddot{R}+\frac{3}{2}\dot{R}^2=\frac{1}{\rho}\left(p_{(R)}-p_\infty\right) \qquad Eq.(1)$$

To determine $p(R)$, we express the mechanical equilibrium at the bubble interface as follows:

$$-\sigma_{rr}*\pi R^2+S*2\pi R=p_{inside}*\pi R^2$$

$$-\sigma_{rr}=p_b-\frac{2S}{R}$$

Where σ_{rr} is the normal stress in the moving liquid at the interface and p_b the pressure in the bubble. In spherical coordinates, this term reads:

$$\sigma_{rr}=-p+2\mu\frac{\partial u_r}{\partial r} \qquad Eq.(2)$$

We need to calculate this term at $r=R$, that is the interface:

$$\sigma_{rr,\ (at\ r=R)}=-p_{(R)}+2\mu\frac{\partial}{\partial r}\left(\frac{R^2\dot{R}}{r^2}\right)=-p_{(R)}-\frac{4\mu\dot{R}}{R}$$

Substituting in Eq. (2):

$$p(R) + \frac{4\mu\dot{R}}{R} = p_b - \frac{2S}{R}$$

Therefore $p(R)$ reads:

$$p(R) = -\frac{2S}{R} - \frac{4\mu\dot{R}}{R} + p_b$$

Substituting in Eq. (1):

$$R\ddot{R} + \frac{3}{2}\dot{R}^2 = \frac{1}{\rho}\left(p_b - \frac{2S}{R} - \frac{4\mu\dot{R}}{R} + p_\infty\right)$$

$$R\ddot{R} + \frac{3}{2}\dot{R}^2 + \frac{4\nu\dot{R}}{R} + \frac{2S}{\rho R} = \frac{1}{\rho}(p_b - p_\infty)$$

Which is the Rayleigh-Plesset equation.

2.2)

Assumptions:

$$\nu = 0 \quad , \quad S = 0 \quad , \quad p_g = 0 \qquad \rightarrow \qquad p_b = p_g + p_v = p_v$$

Therefore, the Rayleigh equation becomes:

$$R\ddot{R} + \frac{3}{2}\dot{R}^2 = \frac{1}{\rho}(p_v - p_\infty)$$

Multiplying the two sides by $2R^2\dot{R}$, we have

$$\rho\left(R\ddot{R} + \frac{3}{2}\dot{R}^2\right)\left(2R^2\dot{R}\right) = (p_v - p_\infty)\left(2R^2\dot{R}\right)$$

$$\rightarrow \qquad \rho\left(2R^3\dot{R}\ddot{R} + 3R^2\dot{R}^3\right) = (p_v - p_\infty)\left(2R^2\dot{R}\right)$$

The left-hand side can be written as:

$$\rho \frac{d}{dt}\left(R^3 \dot{R}^2\right) = (p_v - p_\infty)\left(2R^2 \dot{R}\right)$$

$$\rightarrow \qquad \rho d\left(R^3 \dot{R}^2\right) = (p_v - p_\infty)(2R^2 dR)$$

Taking integral $\int_{R_0}^{R}(\ldots)dR$,

$$\rho\left(R^3 \dot{R}^2\right)_{R_0}^{R} = \frac{1}{3}(R^3 - R_0^3)(p_v - p_\infty)$$

Since $\dot{R}$ is equal to zero for $R = R_0$ (initial condition), we have:

$$\rho\left(R^3 \dot{R}^2\right) = \frac{1}{3}(R^3 - R_0^3)(p_v - p_\infty)$$

Therefore,

$$\dot{R}^2 = \frac{(p_v - p_\infty)}{\rho} * \frac{2}{3}\left(\frac{R^3 - R_0^3}{R^3}\right)$$

$$\dot{R} = -\sqrt{\frac{2}{3}\frac{(p_\infty - p_v)}{\rho}\left(\frac{R_0^3}{R^3} - 1\right)} \qquad Eq. (3)$$

The negative sign is due to the direction of the motion (inward motion).

We use Eq 3 to derive the collapse time as follows:

$$\dot{R} = \frac{dR}{dt} \qquad \rightarrow \qquad dR = \left(-\sqrt{\frac{2}{3}\frac{(p_\infty - p_v)}{\rho}\left(\frac{R_0^3}{R^3} - 1\right)}\right)dt$$

Where R in the right-hand side is a function of time.

$$\frac{dR}{\sqrt{\frac{2}{3}\frac{(p_\infty - p_v)}{\rho}\left(\frac{R_0^3}{R^3} - 1\right)}} = -dt$$

Taking integral from two sides:

$$\int_{0}^{T_i} -dt = \int_{R_0}^{0} \frac{dR}{\sqrt{\frac{2}{3}\frac{(p_\infty - p_v)}{\rho}\left(\frac{R_0^3}{R^3} - 1\right)}}$$

Then, we have:

$$T_i = -\sqrt{\frac{3}{2}\frac{\rho}{(p_\infty - p_v)}} * \int_{R_0}^{0} \frac{dR}{\sqrt{\left(\frac{R_0^3}{R^3} - 1\right)}}$$

By changing the variable from R to λ, by $R = R_0\lambda^2$, we will have:

$$T_i = -R_0\sqrt{\frac{6\rho}{(p_\infty - p_v)}} * \int_{0}^{1} \frac{\lambda^4 d\lambda}{(1-\lambda^6)^{0.5}}$$

The integration on the right-hand side may be solved numerically, which gives:

$$T_i = 0.915 * R_0\sqrt{\frac{\rho}{p_\infty - p_v}}$$

2.3) See attached publication (Obreschkow et. al, 2011)
Be careful that in this document the non-dimensional radius of the bubble is called *r*, which is used hereafter to designate an arbitrary location in the liquid phase.

2.4) See attached publication (Obreschkow et. al, 2011)

2.5) Must be solved numerically (e.g. ode45 function of Matlab)
2.5.a) A bubble filled with vapor only (Rayleigh model)

2.5.b) A bubble filled with vapor and non-condensable gas of a given partial pressure.

2.6) Must be solved numerically (e.g. ode45 function of Matlab), as in 2.5

2.7)

2.7.a) Rayleigh case (bubble filled with vapor only):

$$p_{g_0} = 0 \quad \rightarrow \quad p_{bubble} = p_{vapor} \quad \rightarrow \quad T = T_{sat\,@\,p_v}$$

In this case:

- There is no increase of the gas temperature since it the vapor condenses instantaneously.
- The interface velocity tends to infinity as the bubble radius tends to zero (See eq. 3 above)

2.7.b) Rayleigh-Plesset (bubble filled with vapor and non-condensable gas at partial pressure p_{g_0}:

$$p_g = p_{g_0}\left(\frac{R_0}{R(t)}\right)^{3n}$$

If isothermal process: $n = 1, \quad T = T_0 = T_{sat\,@\,p_v}$

If adiabatic process: $n = \gamma = \frac{c_p}{c_v}, \qquad for\ air: \gamma = 1.4$

$$T_g = T_0\left(\frac{R_0}{R(t)}\right)^{3(\gamma-1)}$$

Here, we need to solve the problem numerically to determine $R(t), T_g(t)$ and $\dot{R}(t)$

Due to the presence of non-condensable gas, the interface velocity is bounded and may be determined numerically.

2.7.c) Keller-Miksis case, adiabatic process is assumed, so

$$p_g = p_{g_0}\left(\frac{R_0}{R(t)}\right)^{3\gamma} \quad \rightarrow \quad T_g = T_0\left(\frac{R_0}{R(t)}\right)^{3(\gamma-1)}$$

As for the Rayleigh-Plesset model, we need to solve the problem numerically to determine $R(t)$, $T_g(t)$ and $\dot{R}(t)$.

2.8)

Requires to be solved numerically (e.g. ode45 function of Matlab)

2.9)

Pressure build-up during the bubble collapse:

Here we write the mass and momentum conservation as in 2.1

$$2R\dot{R}^2r^{-2} + \ddot{R}R^2r^{-2} - 2R^4\dot{R}^2r^{-5} = -\frac{1}{\rho}\frac{\partial p}{\partial r}$$

To obtain the pressure in the liquid during the bubble collapse, we must integrate the momentum equation between an arbitrary radius r>R(t) to infinity ($\int_r^{\infty}(\dots)dr$). This results in the following:

$$\frac{p_\infty - p_{(r)}}{\rho} = -\frac{\ddot{R}R^2}{r} + 2\dot{R}^2\left(\frac{R^4}{4R^4} - \frac{R}{r}\right)$$

Using the following equations, already established earlier:

$$\begin{cases} \dot{R}^2 = \dfrac{(p_v - p_\infty)}{\rho} * \dfrac{2}{3}\left(\dfrac{R^3 - R_0^3}{R^3}\right) \\ \ddot{R} = -\dfrac{1}{R}\left(\dfrac{p_\infty - p_v}{\rho} + \dfrac{3}{2}\dot{R}^2\right) \end{cases}$$

We obtain the following equation:

$$\frac{p_\infty - p_{(r)}}{p_\infty - p_v} = \frac{R}{3r}\left(\frac{R_0^3}{R^3} - 4\right) - \frac{R^4}{3r^4}\left(\frac{R_0^3}{R^3} - 1\right)$$

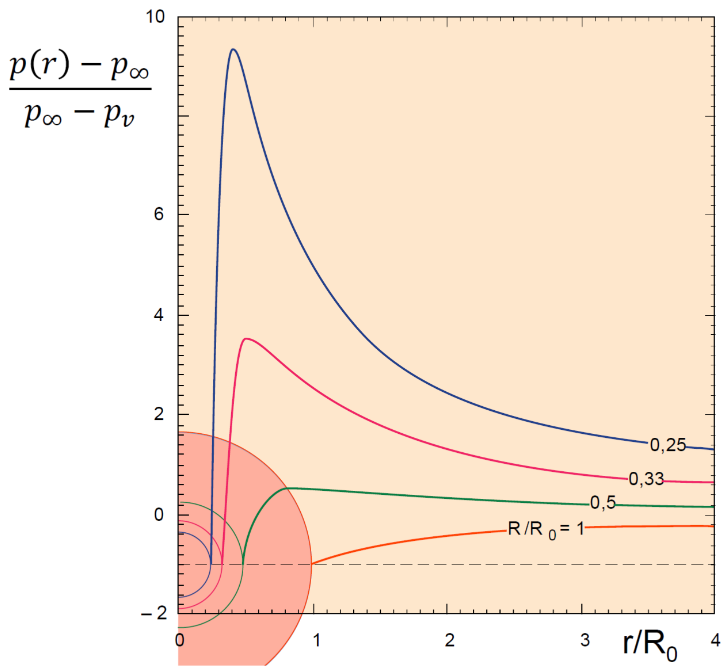
To find the maximum pressure in the field and its location, it is required to take the derivative of the above equation with respect to *r*. Doing so, we will have:

$$r_{max} = R\left(4\frac{1-\frac{R^3}{R_0^3}}{1-4\frac{R^3}{R_0^3}}\right)^{\frac{1}{3}}$$

And

$$\frac{p_{max}-p_\infty}{p_\infty-p_v} = \frac{1}{4^{4/3}}\frac{\left(1-4\frac{R^3}{R_0^3}\right)^{4/3}}{\frac{R^3}{R_0^3}\left(1-\frac{R^3}{R_0^3}\right)^{1/3}}$$

The corresponding plot:



2.10)

Owing to mass conservation, we have the following relation for the velocity u_r everywhere in the liquid phase:

$$u_r = \frac{R^2\dot{R}}{r^2}$$

Which may be re-written using Eq. 3 as follows:

$$u_r = -\frac{R^2}{r^2}\sqrt{\frac{2}{3}\frac{(p_\infty - p_v)}{\rho}\left(\frac{R_0^3}{R^3} - 1\right)} = -\frac{1}{r^2}\sqrt{\frac{2}{3}\frac{(p_\infty - p_v)}{\rho}}\sqrt{RR_0^3 - R^4}$$

u_r is a function of the bubble radius, which depends on time. Since the bubble does not rebound (bubble filled with vapor only), the liquid must fill the gap left by the collapsing bubble. To do so, the liquid everywhere is accelerated from rest to a maximum absolute value. It is then decelerated to zero velocity when the bubble completes its collapse.

For a given location (r), the bubble radius (R_1) corresponding to the occurrence of maximum absolute velocity in the liquid is determined by solving ($\frac{\partial u_r}{\partial R} = 0$):

$$\frac{\partial u_r}{\partial R} = 0 \quad \rightarrow \quad R_0^3 - 4R^3 = 0 \quad \rightarrow \quad R_1 = \sqrt[3]{\frac{1}{4}}R_0 \approx 0.63R_0$$

Owing to Eq. 3, the corresponding interface velocity, $\dot{R}_1$, is :

$$\dot{R}_1 = -\sqrt{\frac{2}{3}\frac{(p_\infty - p_v)}{\rho}\left(\frac{R_0^3}{{R_1}^3} - 1\right)} = -\sqrt{2\frac{(p_\infty - p_v)}{\rho}}$$

And the maximum velocity, u_{r_max}, at any arbitrary location (r) is given by the mass conservation:

$$u_{r\_max} = \frac{{R_1}^2\dot{R}_1}{r^2} = -\frac{{R_1}^2}{r^2}\sqrt{2\frac{(p_\infty - p_v)}{\rho}} = -\frac{2^{\frac{1}{2}}}{4^{\frac{2}{3}}}\sqrt{\frac{(p_\infty - p_v)}{\rho}}\frac{{R_0}^2}{r^2}$$
$$= -2^{\frac{-5}{6}}\sqrt{\frac{(p_\infty - p_v)}{\rho}}\frac{{R_0}^2}{r^2}$$

We may notice that:

- the maximum velocity is reached at the same time in the entire liquid phase. This is a consequence of the incompressibility assumption.

- $\lim_{r\to\infty}\left(u_{r\_max}\right) = 0$

1.