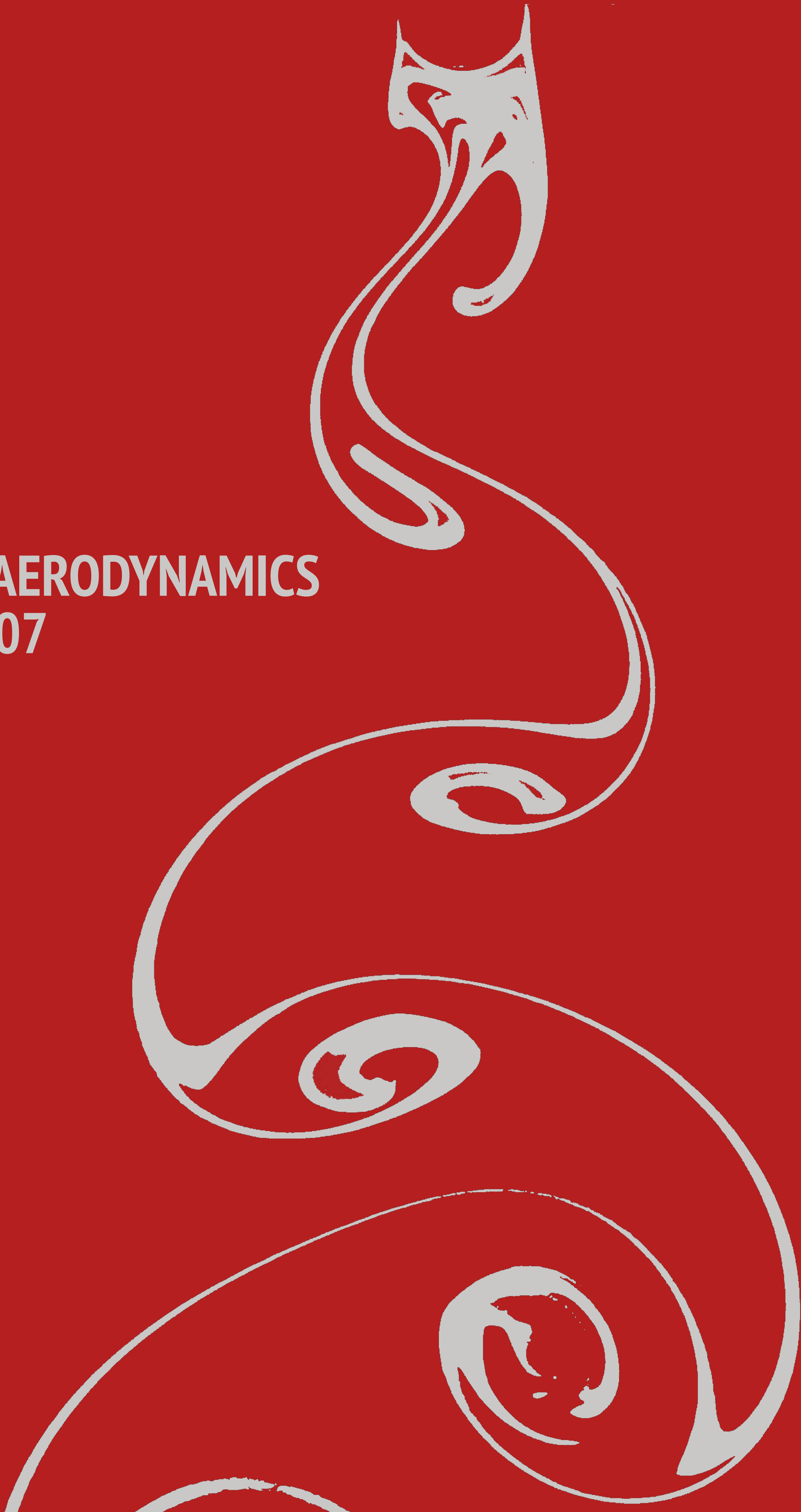
ME-445 AERODYNAMICS
Exercise 07
Week 7



Formula sheet

Cylindrical coordinates

$$\nabla\vec{u} = \left(\frac{\partial v_r}{\partial r}, \frac{1}{r}\frac{\partial v_\theta}{\partial \theta}, 0\right)$$

$$\nabla \cdot \vec{u} = \frac{1}{r}\frac{\partial (r v_r)}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta}$$

$$\nabla \times \vec{u} = \left(0, 0, \frac{1}{r}\left[\frac{\partial (r v_\theta)}{\partial r} - \frac{\partial v_r}{\partial \theta}\right]\right)$$

Potential flow

$$v_r = \frac{\partial \phi}{\partial r} = \frac{1}{r}\frac{\partial \psi}{\partial \theta}, \quad v_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = -\frac{\partial \psi}{\partial r}$$

Uniform parallel flow $\quad w = \phi + i\psi = U_\infty e^{-i\alpha}\, z$

Potential vortex in z_0 $\quad w = -\dfrac{i\gamma}{2\pi}\ln(z - z_0)$

Point source or sink in z_0 $\quad w = \dfrac{Q}{2\pi}\ln(z - z_0)$

Source-sink doublet in z_0 $\quad w = \dfrac{\mu}{2\pi(z - z_0)}$

$$\frac{\mathrm{d}w}{\mathrm{d}z} = u - iv$$

Milne-Thomson circle theorem:

$$g(z) = w(z) + \overline{w\left(\frac{a^2}{\bar{z}}\right)}$$

Thin airfoil theory

For a camber line with:

$$\frac{\mathrm{d}y_c}{\mathrm{d}x} = A_0 + \sum_{n=1}^{\infty} A_n \cos n\theta$$

$$\frac{x}{c} = \frac{(1 - \cos\theta)}{2}$$

we know:

$$k = 2U_\infty\left[(\alpha - A_0)\frac{\cos\theta + 1}{\sin\theta} + \sum_{n=1}^{\infty} A_n \sin n\theta\right]$$

$$A_0 = \frac{1}{\pi}\int_0^\pi \frac{\mathrm{d}y_c}{\mathrm{d}x}\mathrm{d}\theta$$

$$A_n = \frac{2}{\pi}\int_0^\pi \frac{\mathrm{d}y_c}{\mathrm{d}x}\cos n\theta \mathrm{d}\theta$$

$$C_l = 2\pi\alpha + \pi(A_1 - 2A_0)$$

$$C_{m,1/4} = -\frac{\pi}{4}(A_1 - A_2)$$

$$x_{cp} = \frac{1}{4} + \frac{\pi}{4C_l}(A_1 - A_2)$$

Finite wings with AR=b^2/S

Sign convention:

if induced velocity points downward: $w(y) > 0$, $\alpha_i(y) > 0$

if induced velocity points upward: $w < 0$, $\alpha_i < 0$

Prandtl's lifting-line theory

$$U_\infty \alpha_i(y_0) = w(y_0) = -\frac{1}{4\pi}\int_{-b/2}^{b/2} \frac{(\mathrm{d}\Gamma/\mathrm{d}y)}{y - y_0}\mathrm{d}y$$

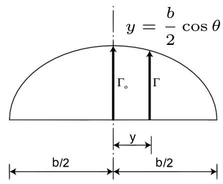
$$\alpha(y_0) = \alpha_{eff}(y_0) + \alpha_i(y_0)$$

Elliptical loading $\Gamma(y) = \Gamma_0\sqrt{1 - \left(\dfrac{2y}{b}\right)^2}$

$$w = \frac{\Gamma_0}{2b}$$

$$\alpha_i = \frac{C_L}{\pi \mathrm{AR}}$$

$$C_{D,i} = \frac{C_L^2}{\pi \mathrm{AR}}$$



General loading $\Gamma(\theta) = 2bU_\infty \sum_{n=1}^{\infty} A_n \sin n\theta$

$$w(\theta) = U_\infty \sum_{n=1}^{\infty} n A_n \frac{\sin n\theta}{\sin\theta}$$

$$C_L = \pi A_1 \mathrm{AR}$$

$$C_{D,i} = \frac{C_L^2}{\pi \mathrm{AR}}(1 + \delta) \text{ with } \delta = \sum_{n=2}^{\infty} n\,(A_n/A_1)^2$$

Boundary Layer

Flat plate **laminar** boundary layer

$\dfrac{\delta}{x} = \dfrac{5}{\sqrt{Re_x}}$ boundary layer growth

$C_f = \dfrac{1.328}{\sqrt{Re_x}}$ skin friction drag coefficient

Flat plate **turbulent** boundary layer

$\dfrac{\delta}{x} = \dfrac{0.37}{Re_x^{1/5}}$ boundary layer growth

$C_f = \dfrac{0.074}{Re_x^{1/5}}$ skin friction drag coefficient

Miscellanous

θ	0°	30°	45°	60°	90°
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0

water

kinematic viscosity $\nu = 1 \times 10^{-6}\,\mathrm{m}^2\,\mathrm{s}^{-1}$

density $\rho = 1000\,\mathrm{kg}\,\mathrm{m}^{-3}$

air

kinematic viscosity $\nu = 1.5 \times 10^{-5}\,\mathrm{m}^2\,\mathrm{s}^{-1}$

density $\rho = 1.2\,\mathrm{kg}\,\mathrm{m}^{-3}$

$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$$
$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$$

$$\cos 2\theta = 2\cos^2\theta - 1$$
$$\sin 2\theta = 2\sin\theta\cos\theta$$
$$\sin 3\theta = 3\sin\theta - 4\sin^3\theta$$
$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$$

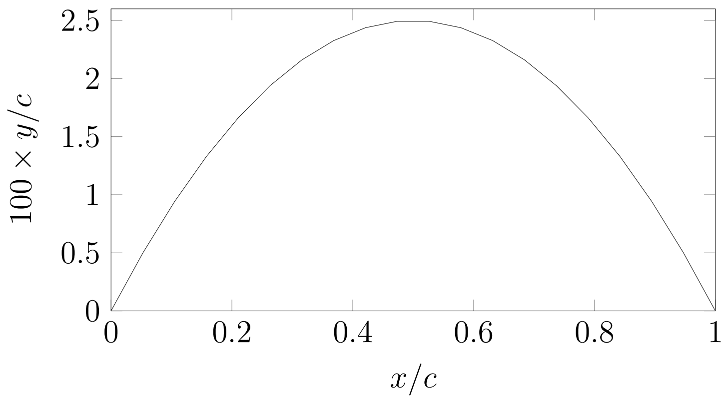
$$\int_0^\pi \cos\theta \mathrm{d}\theta = 0$$
$$\int_0^\pi \sin\theta \mathrm{d}\theta = 2$$
$$\int_0^\pi \cos^2\theta \mathrm{d}\theta = \int_0^\pi \sin^2\theta \mathrm{d}\theta = \frac{\pi}{2}$$

$$\int_0^\pi \frac{\cos n\theta}{\cos\theta - \cos\theta_1}\mathrm{d}\theta = \pi\,\frac{\sin n\theta_1}{\sin\theta_1} \qquad n = 0, 1, 2, \ldots$$
$$\int_0^\pi \frac{\sin n\theta \sin\theta}{\cos\theta - \cos\theta_1}\mathrm{d}\theta = -\pi\cos n\theta_1 \qquad n = 1, 2, 3, \ldots$$

1. We want to design a thin airfoil with a specific amount of camber. The camber line is approximated by $\frac{y_c}{c} = a\left[\frac{1}{4} - \left(\frac{x}{c} - \frac{1}{2}\right)^2\right]$, with a a positive constant.

   (a) Draw a sketch of this airfoil

   Solution:



   (b) What is the value of the parameter a if we want the airfoil to have $2.5\,\%$ camber?

   Solution: First, compute the derivative of the camber line:

$$\frac{dy_c}{dx} = -a\left(\frac{2x}{c} - 1\right)$$

Thus, the maximum camber occurs at $\xi = 0.5$, where $\xi = x/c$, and its value is given by:

$$\left.\frac{y_c}{c}\right|_{\xi=0.5} = 0.25a$$

The maximum camber will be 2.5% for $a = 0.1$.

   (c) Determine the coefficients A_0, A_1 and A_2.

   Solution: From the previous question:

$$\frac{1}{c}\frac{dy_c}{dx} = -a\left(2\xi - 1\right) \tag{1}$$

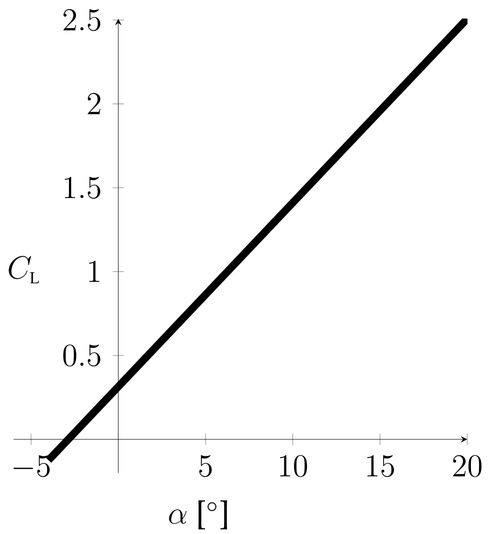
Using the transformation $\xi = \frac{1-\cos(\theta)}{2}$, this expression can be written as:

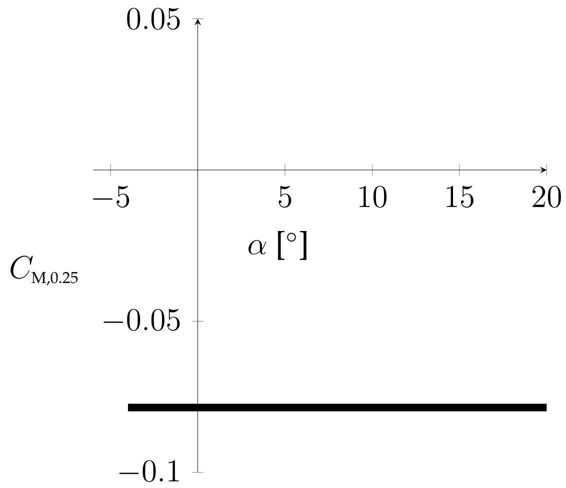
$$\frac{dy_c}{dx} = a\cos(\theta)$$

Comparing this expression to that in the formula sheet, we deduce that: $A_0 = A_2 = 0$, $A_1 = a$.

   (d) Draw the lift and quarter chord moment coefficients, C_L and $C_{M,1/4}$ in function of α.

Solution:





(e) What is the angle of attack for zero lift? And what are C_l and $C_{m,1/4}$ at $\alpha = 0$?

Solution: $C_l(\alpha = -0.05 = -2.86°) = 0$
$C_l(\alpha = 0) = a\pi = 0.314$
$C_{m,1/4}(\alpha = 0) = -0.025\pi = -0.0785$

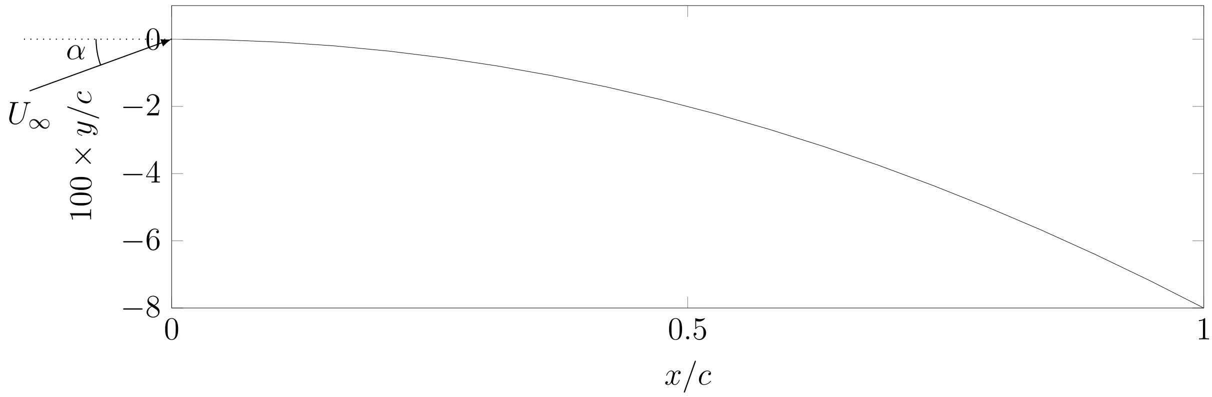
(f) Define and calculate the aerodynamic centre and the centre of pressure for this airfoil at zero angle of attack?

Solution: The aerodynamic center is a point on the camber line, generally near quarter chord where the pitching moment is independent of the angle of attack. The center of pressure is a point on the camber line where the pitching moment is zero; it depends on the angle of attack. $x_{cp} = 1/2$
$\xi_{AC} = 1/4$

(g) Can you think of a way to change the airfoil to reduce the moment while keeping the maximum camber the same?

Solution: Shift the point of maximum camber to the LE.

2. Thin airfoil theory is used to describe the two-dimensional potential flow around a parabolically curved thin plate of length L place in a uniform free stream with velocity U_∞ at an angle of attack α, as shown in the figure below.



The plate shape is given by:

$$\frac{y_c}{c} = -0.08 \left(\frac{x}{c}\right)^2$$

(a) Calculate the Fourier coefficients A_0, A_1 and A_2 for this camber line.

Solution: First, compute the camber line derivative and use the transformation $\frac{x}{c} = \frac{1-\cos(\theta)}{2}$:

$$\frac{dy_c}{dx} = -0.08 \times 2\frac{x}{c} = 0.08(\cos(\theta) - 1)$$

Comparing this expression to the one in the formula sheet, we have $A_0 = -0.08$ and $A_1 = 0.08$. The following Fourier coefficients are null.

(b) Determine the lift coefficient and position of the center of pressure for $\alpha = 0$.

Solution: From the Fourier coefficients, we can calculate the lift coefficient and position of the center of pressure as follows:

$$C_l = 2\pi\alpha + \pi(A_1 - 2A_0)$$
$$x_{cp} = \frac{1}{4} + \frac{\pi}{4C_l}(A_1 - A_2)$$

for $\alpha = 0$ that yields:

$$C_l = 0.24\pi$$
$$x_{cp} = \frac{1}{3}$$

(c) Determine the value of α for which the pressure difference between the upper and lower surface of the plate is zero at the leading edge (no suction at the leading edge).

Solution: A zero pressure difference implies that the velocity above and below the airfoil must be the same. In thin-airfoil theory, the airfoil is replace by a vortex sheet. In this case, the strength of the vortex sheet must be zero at the leading edge. The chordwise circulation distribution is given by:

$$k = 2\mathrm{U}_\infty \left[(\alpha - A_0) \frac{\cos\theta + 1}{\sin\theta} + \sum_{n=1}^{\infty} A_n \sin n\theta \right]$$

For the airfoil in this problem, that reduces to:

$$k = 2\mathrm{U}_\infty \left[(\alpha - A_0) \frac{\cos\theta + 1}{\sin\theta} + A_1 \sin(\theta) \right]$$

Applying the boundary condition $k(\theta = 0) = 0$, we deduce that $A_0 = \alpha = -4.6°$. This only works if you ignore the indeterminacy that occurs by plugging in $(\theta = 0$ since that causes you to divide by 0.

3. In this problem, we will investigate the aerodynamic performance of a thin airfoil that has a camber line defined by a third order polynomial $y_c = bc(\xi - a_1)(\xi - a_2)(\xi - a_3)$, where ξ is the dimensionless chord position.

   (a) Determine the values of a_1 and a_2 so that the polynomial function describes a real camber-line ($y_c = 0$ at $x = 0$ and $x = c$) and thus show that it can be described by the following equation: $y_c = bc\xi(\xi - 1)(\xi - a)$. What do the remaining parameters b and a represent in terms of airfoil geometry?

   Solution: Given that $y_c = 0$ at $x = 0$ and $x = c$, thus $\xi = 0$ and $\xi = 1$, we know the roots of the polynomial defining y_c are $a_1 = 0$ and $a_2 = 1$. Therefore:

   $$y_c = bc\xi(\xi - 1)(\xi - a)$$

   From this expression, we can deduce b defines maximum camber and a sets the position of the airfoil inflection point.

   (b) Making use of the change in variable $\xi = \dfrac{x}{c} = \dfrac{1 - \cos\theta}{2}$, show that the camberline derivative can be written as:

   $$\frac{dy_c}{dx} = b\left[\frac{1}{8} + \left(a - \frac{1}{2}\right)\cos\theta + \frac{3}{8}\cos 2\theta\right]$$

   Solution: We will use the following transformations to change the camber-line variable from cartesian to azimuthal. This will allow us to determine the Fourier series coefficients.

$$\begin{cases} \dfrac{x}{c} = \xi = \dfrac{1 - \cos\theta}{2} \\ \xi^2 = \dfrac{1 - 2\cos\theta + \cos^2\theta}{4} \end{cases}$$

$$\begin{aligned} y_c &= bc(\xi^3 - (a+1)\xi^2 + a) \\ \frac{dy_c}{dx} = \frac{dy_c}{d\xi \cdot c} &= \frac{1}{c}bc(3\xi^2 - 2(a+1)\xi + a) \\ &= b\left[3\Big(\frac{1 - 2\cos\theta + \cos^2\theta}{4}\Big) - 2(a+1)\Big(\frac{1 - \cos\theta}{2}\Big) + a\right] \\ &= b\left[\frac{3}{4} - (a+1) + a - \frac{3}{2}\cos\theta + (a+1)\cos\theta + \frac{3}{4}\cos^2\theta\right] \end{aligned}$$

   Using $2\cos^2\theta = 1 + \cos 2\theta$

$$\begin{aligned} &= b\left[\frac{3}{4} - 1 + \frac{3}{8} + \left(a - \frac{1}{2}\right)\cos\theta + \frac{3}{4}\Big(\frac{1 + \cos 2\theta}{2}\Big)\right] \\ &= b\left[\frac{1}{8} + \left(a - \frac{1}{2}\right)\cos\theta + \frac{3}{8}\cos 2\theta\right] \end{aligned}$$

(c) Show that the Fourier coefficients for a third order polynomial camberline are given by:

$$\begin{cases} A_0 = \frac{b}{8} \\ A_1 = \Big(a - \frac{1}{2}\Big)b \\ A_2 = \frac{3b}{8} \end{cases}$$

Solution:

Fourier series are given by $\dfrac{dy_c}{dx} = A_0 + \sum\limits_{n=1}^{\infty} A_n \cos n\theta$, thus by comparing this expression to that found in the previous part:

$$\begin{cases} A_0 = \frac{b}{8} \\ A_1 = \Big(a - \frac{1}{2}\Big)b \\ A_2 = \frac{3b}{8} \end{cases}$$

(d) Show that the coefficients of lift and pitching moment (C_l and $C_m|_{AC}$) for an airfoil whose camber line is defined by a third order polynomial at an angle of attack α are given by:

$$\begin{cases} C_l = 2\pi\alpha + \pi b\Big(a - \dfrac{3}{4}\Big) \\ C_m|_{1/4} = -\dfrac{\pi}{4}b\Big(a - \dfrac{7}{8}\Big) \end{cases}$$

Solution:

For a general thin airfoil:

$$\begin{cases} C_l = 2\pi\alpha + \pi(A_1 - 2A_0) = 2\pi\alpha + \pi b\Big(a - \dfrac{3}{4}\Big) \\ C_m|_{LE} = -\dfrac{C_l}{4}\Big(1 + \dfrac{A_1 - A_2}{C_l/\pi}\Big) \\ C_m|_{AC} = C_m|_{LE} + C_l\Big(\dfrac{1}{4} - 0\Big) = -\dfrac{\pi}{4}(A_1 - A_2) = -\dfrac{\pi}{4}b\Big(a - \dfrac{7}{8}\Big) \end{cases}$$

(e) For an airfoil with $a = 2$ and a maximum camber of 2%, show that $b = 0.052$ and determine the coefficients of lift and pitching moment (C_l and $C_{m,1/4}$) at a three degree angle of attack.

Solution:

Maximum camber occurs at

$$\begin{aligned} \frac{dy_c}{d\xi} &= 0 \\ 3\xi^2 - 6\xi + 2 &= 0 \\ \xi_{max} &= 0.42 \end{aligned}$$

We know that the maximum camber is $\dfrac{y_{c_{max}}}{c} = 2\%$

$$\begin{aligned} \frac{y_c(\xi = \xi_{max})}{c} = 0.38\, b &= 0.02 \\ b &= 0.052 \end{aligned}$$

Taking $a = 2$, $b = 0.052$ and $\alpha = \dfrac{3\pi}{180}$ we find:

$$\begin{cases} C_l = 2\pi\alpha + \pi b\Big(a - \dfrac{3}{4}\Big) = 0.533 \\ C_m|_{1/4} = -\dfrac{\pi}{4} b\Big(a - \dfrac{7}{8}\Big) = -0.046 \end{cases}$$