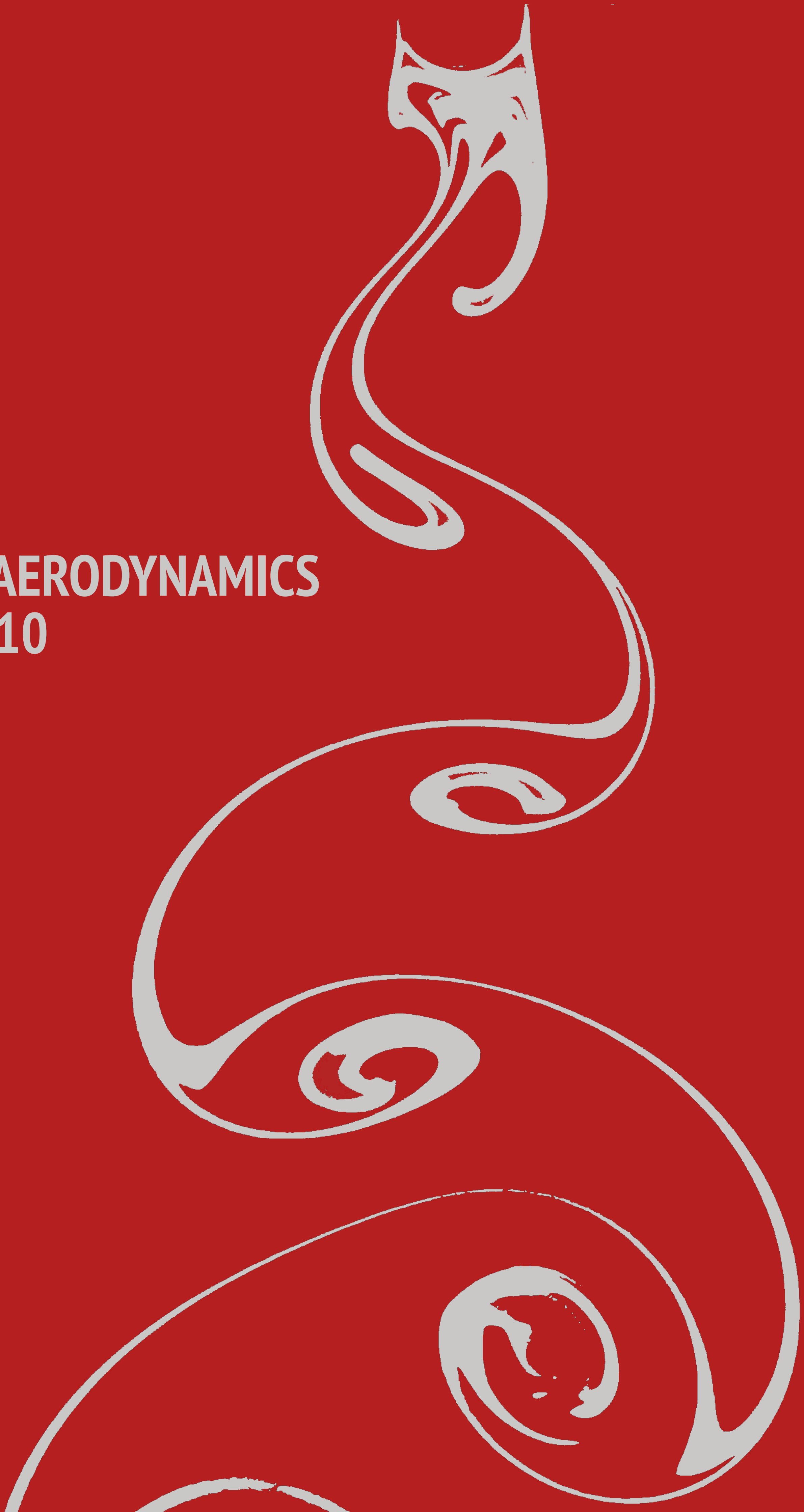
ME-445 AERODYNAMICS
Exercise 10
Week 11



Formula sheet

Cylindrical coordinates

$$\nabla\vec{u} = \left(\frac{\partial v_r}{\partial r}, \frac{1}{r}\frac{\partial v_\theta}{\partial \theta}, 0\right)$$

$$\nabla\cdot\vec{u} = \frac{1}{r}\frac{\partial (r v_r)}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta}$$

$$\nabla\times\vec{u} = \left(0, 0, \frac{1}{r}\left[\frac{\partial (r v_\theta)}{\partial r} - \frac{\partial v_r}{\partial \theta}\right]\right)$$

Potential flow

$$v_r = \frac{\partial \phi}{\partial r} = \frac{1}{r}\frac{\partial \psi}{\partial \theta}, \quad v_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = -\frac{\partial \psi}{\partial r}$$

Uniform parallel flow $\quad w = \phi + i\psi = U_\infty e^{-i\alpha}\, z$

Potential vortex in z_0 $\quad w = -\frac{i\gamma}{2\pi}\ln(z - z_0)$

Point source or sink in z_0 $\quad w = \frac{Q}{2\pi}\ln(z - z_0)$

Source-sink doublet in z_0 $\quad w = \frac{\mu}{2\pi(z - z_0)}$

$$\frac{\mathrm{d}w}{\mathrm{d}z} = u - iv$$

Milne-Thomson circle theorem:

$$g(z) = w(z) + \overline{w\left(\frac{a^2}{\bar{z}}\right)}$$

Thin airfoil theory

For a camber line with:

$$\frac{\mathrm{d}y_c}{\mathrm{d}x} = A_0 + \sum_{n=1}^{\infty} A_n \cos n\theta$$

$$\frac{x}{c} = \frac{(1 - \cos\theta)}{2}$$

we know:

$$k = 2U_\infty\left[(\alpha - A_0)\frac{\cos\theta + 1}{\sin\theta} + \sum_{n=1}^{\infty} A_n \sin n\theta\right]$$

$$A_0 = \frac{1}{\pi}\int_0^\pi \frac{\mathrm{d}y_c}{\mathrm{d}x}\mathrm{d}\theta$$

$$A_n = \frac{2}{\pi}\int_0^\pi \frac{\mathrm{d}y_c}{\mathrm{d}x}\cos n\theta \mathrm{d}\theta$$

$$C_l = 2\pi\alpha + \pi(A_1 - 2A_0)$$

$$C_{m,1/4} = -\frac{\pi}{4}(A_1 - A_2)$$

$$x_{cp} = \frac{1}{4} + \frac{\pi}{4C_l}(A_1 - A_2)$$

Finite wings with AR=b^2/S

Sign convention:

if induced velocity points downward: $w(y) > 0$, $\alpha_i(y) > 0$

if induced velocity points upward: $w < 0$, $\alpha_i < 0$

Prandtl's lifting-line theory

$$U_\infty \alpha_i(y_0) = w(y_0) = -\frac{1}{4\pi}\int_{-b/2}^{b/2} \frac{(\mathrm{d}\Gamma/\mathrm{d}y)}{y - y_0}\mathrm{d}y$$

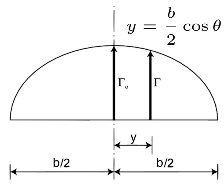
$$\alpha(y_0) = \alpha_{\text{eff}}(y_0) + \alpha_i(y_0)$$

Elliptical loading $\Gamma(y) = \Gamma_0\sqrt{1 - \left(\frac{2y}{b}\right)^2}$

$$w = \frac{\Gamma_0}{2b}$$

$$\alpha_i = \frac{C_L}{\pi \mathrm{AR}}$$

$$C_{D,i} = \frac{C_L^2}{\pi \mathrm{AR}}$$



General loading $\Gamma(\theta) = 2bU_\infty \sum_{n=1}^{\infty} A_n \sin n\theta$

$$w(\theta) = U_\infty \sum_{n=1}^{\infty} nA_n \frac{\sin n\theta}{\sin\theta}$$

$$C_L = \pi A_1 \mathrm{AR}$$

$$C_{D,i} = \frac{C_L^2}{\pi \mathrm{AR}}(1 + \delta) \text{ with } \delta = \sum_{n=2}^{\infty} n\left(A_n/A_1\right)^2$$

Boundary Layer

Flat plate **laminar** boundary layer

$\frac{\delta}{x} = \frac{5}{\sqrt{Re_x}}$ boundary layer growth

$C_f = \frac{1.328}{\sqrt{Re_x}}$ skin friction drag coefficient

Flat plate **turbulent** boundary layer

$\frac{\delta}{x} = \frac{0.37}{Re_x^{1/5}}$ boundary layer growth

$C_f = \frac{0.074}{Re_x^{1/5}}$ skin friction drag coefficient

Miscellanous

θ	$0°$	$30°$	$45°$	$60°$	$90°$
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0

water

kinematic viscosity $\nu = 1 \times 10^{-6}\,\mathrm{m}^2\,\mathrm{s}^{-1}$

density $\rho = 1000\,\mathrm{kg}\,\mathrm{m}^{-3}$

air

kinematic viscosity $\nu = 1.5 \times 10^{-5}\,\mathrm{m}^2\,\mathrm{s}^{-1}$

density $\rho = 1.2\,\mathrm{kg}\,\mathrm{m}^{-3}$

$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$$
$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$$

$$\cos 2\theta = 2\cos^2\theta - 1$$
$$\sin 2\theta = 2\sin\theta\cos\theta$$
$$\sin 3\theta = 3\sin\theta - 4\sin^3\theta$$
$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$$

$$\int_0^\pi \cos\theta \mathrm{d}\theta = 0$$
$$\int_0^\pi \sin\theta \mathrm{d}\theta = 2$$
$$\int_0^\pi \cos^2\theta \mathrm{d}\theta = \int_0^\pi \sin^2\theta \mathrm{d}\theta = \frac{\pi}{2}$$

$$\int_0^\pi \frac{\cos n\theta}{\cos\theta - \cos\theta_1}\mathrm{d}\theta = \pi\frac{\sin n\theta_1}{\sin\theta_1} \qquad n = 0, 1, 2, \ldots$$
$$\int_0^\pi \frac{\sin n\theta \sin\theta}{\cos\theta - \cos\theta_1}\mathrm{d}\theta = -\pi\cos n\theta_1 \qquad n = 1, 2, 3, \ldots$$

1. Consider a wing with an aspect ratio $\mathrm{AR} = 10$. Its non-dimensional circulation distribution is

$$\frac{\Gamma(\theta)}{b\mathrm{U}_\infty} = \frac{2\epsilon}{\pi\mathrm{AR}}(3\sin\theta + \sin 3\theta)$$

with $\frac{y}{b} = \frac{1}{2}\cos\theta$, b the wing span from tip to tip, and ϵ a constant value.

(a) Determine the Fourier coefficients according to Prandtl's lifting line theory.

Solution:

$$\begin{aligned}\frac{\Gamma(\theta)}{2b\mathrm{U}_\infty} &= \frac{\epsilon}{\pi\mathrm{AR}}(3\sin\theta + \sin 3\theta)\\ &= \frac{3\epsilon}{\pi\mathrm{AR}}\sin\theta + \frac{\epsilon}{\pi\mathrm{AR}}\sin 3\theta\\ &= A_1\sin\theta + A_3\sin 3\theta\end{aligned}$$

$$\boxed{A_0 = 0, \quad A_1 = \frac{3\epsilon}{\pi\mathrm{AR}}, \quad A_2 = 0, \quad A_3 = \frac{\epsilon}{\pi\mathrm{AR}}, \quad A_n = 0 \text{ for } n \geq 4.}$$

(b) Calculate the lift coefficient C_L <u>and</u> the induced drag coefficient $C_{D,i}$ of this wing.

Solution:

Lift coefficient: $C_L = \pi A_1 \mathrm{AR} = 3\epsilon$

$$\begin{aligned}\delta &= \sum_{n=2}^{\infty} n\left(\frac{A_n}{A_1}\right)^2\\ &= 3\left(\frac{A_3}{A_1}\right)^2\\ &= 3\left(\frac{1}{3}\right)^2 = \frac{1}{3}\end{aligned}$$

Induced drag coefficient: $C_{D,i} = \frac{C_L^2}{\pi\mathrm{AR}}(1+\delta) = \frac{9\epsilon^2}{\pi\mathrm{AR}}\frac{4}{3} = \frac{12\epsilon^2}{\pi\mathrm{AR}}$

$$\begin{aligned}C_L &= 3\epsilon\\ C_{D,i} &= \frac{6}{5\pi}\epsilon^2\end{aligned}$$

(c) Write the expression for the induced angle of attack α_i as a function of θ.

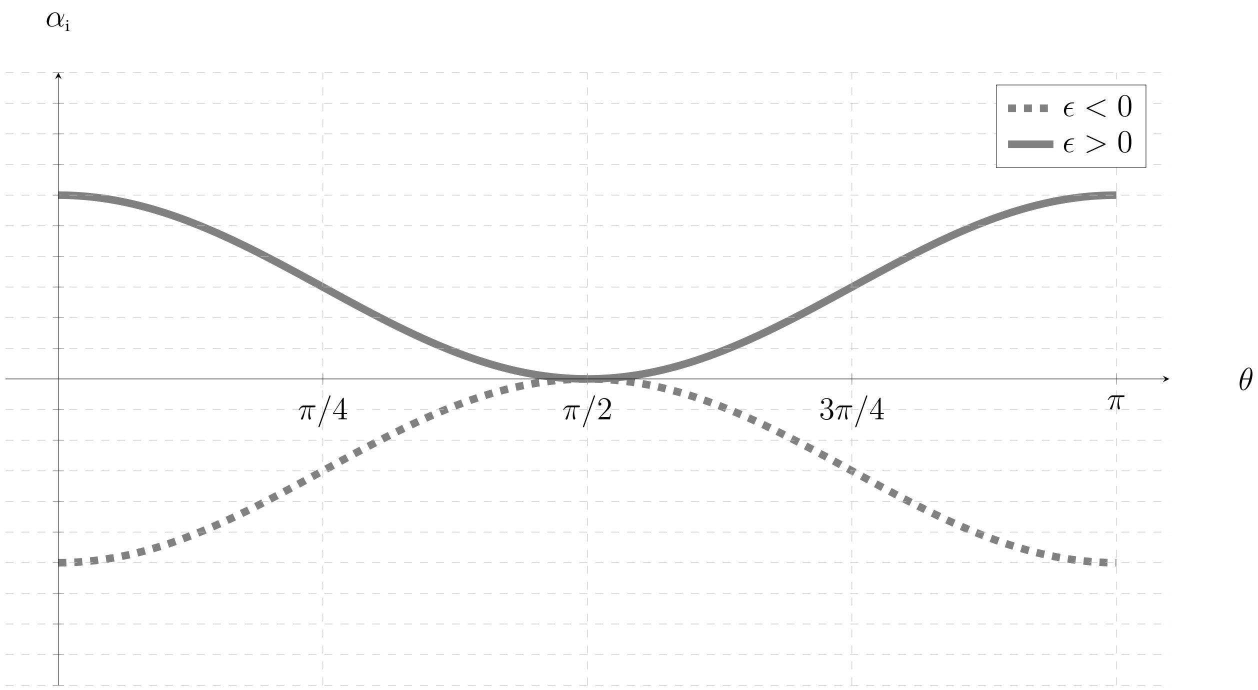
Solution:

$$\begin{aligned}
\alpha_{\mathrm{i}} &= \frac{w(\theta)}{\mathrm{U}_\infty} \\
&= \sum_{n=1}^{\infty} nA_{\mathrm{n}} \frac{\sin n\theta}{\sin\theta} \\
&= A_1 + 3A_3 \frac{\sin 3\theta}{\sin\theta} \\
&= \frac{3\epsilon}{\pi \mathrm{AR}} + 3\frac{\epsilon}{\pi \mathrm{AR}} \frac{\sin 3\theta}{\sin\theta} \\
&= \frac{3\epsilon}{\pi \mathrm{AR}} \left(1 + \frac{\sin 3\theta}{\sin\theta}\right) \\
&= \frac{3\epsilon}{\pi \mathrm{AR}} \left(1 + \frac{3\sin\theta - 4\sin^3\theta}{\sin\theta}\right) \\
&= \frac{12\epsilon}{\pi \mathrm{AR}} \left(1 - \sin^2\theta\right)
\end{aligned}$$

$$\boxed{\alpha_{\mathrm{i}} = \frac{6}{5}\frac{\epsilon}{\pi}\cos^2\theta}$$

(d) Sketch $\alpha_{\mathrm{i}}(\theta)$ for $\epsilon > 0$ and for $\epsilon < 0$. Do not forget to add a legend.

Solution:



(e) The wing is formed by thin airfoils with the same airfoil profiles. At which span-wise location would this wing starts to stall first for $\epsilon > 0$ and where for $\epsilon < 0$?
Motivate your answer.

Solution: The wing is formed by thin airfoils with the same airfoil profiles. The stall angle of attack is the same along the span. Stall begins where the effective angle of attack is highest $\alpha_{\mathrm{eff}} = \alpha - \alpha_{\mathrm{i}}$.

For $\epsilon > 0$, the downwash is positive (i.e. pointing downward) and the induced angle is $\alpha_{\text{i}} \leq 0$ everywhere $\Rightarrow \alpha_{\text{eff}} \leq \alpha$. The effective angle is highest where the induced angle is zero. This occurs at the roots.

For $\epsilon < 0$, the downwash is negative (i.e. pointing upward) and the induced angle is $\alpha_{\text{i}} \geq 0$ everywhere $\Rightarrow \alpha_{\text{eff}} \geq \alpha$. The effective angle is highest where the absolute value of the induced angle is highest. This occurs at the tips.

$\epsilon > 0$: stall begins at the root $\quad \theta = \pi/2$

$\epsilon < 0$: stall begins at the wing tips $\quad \theta = 0, \pi$