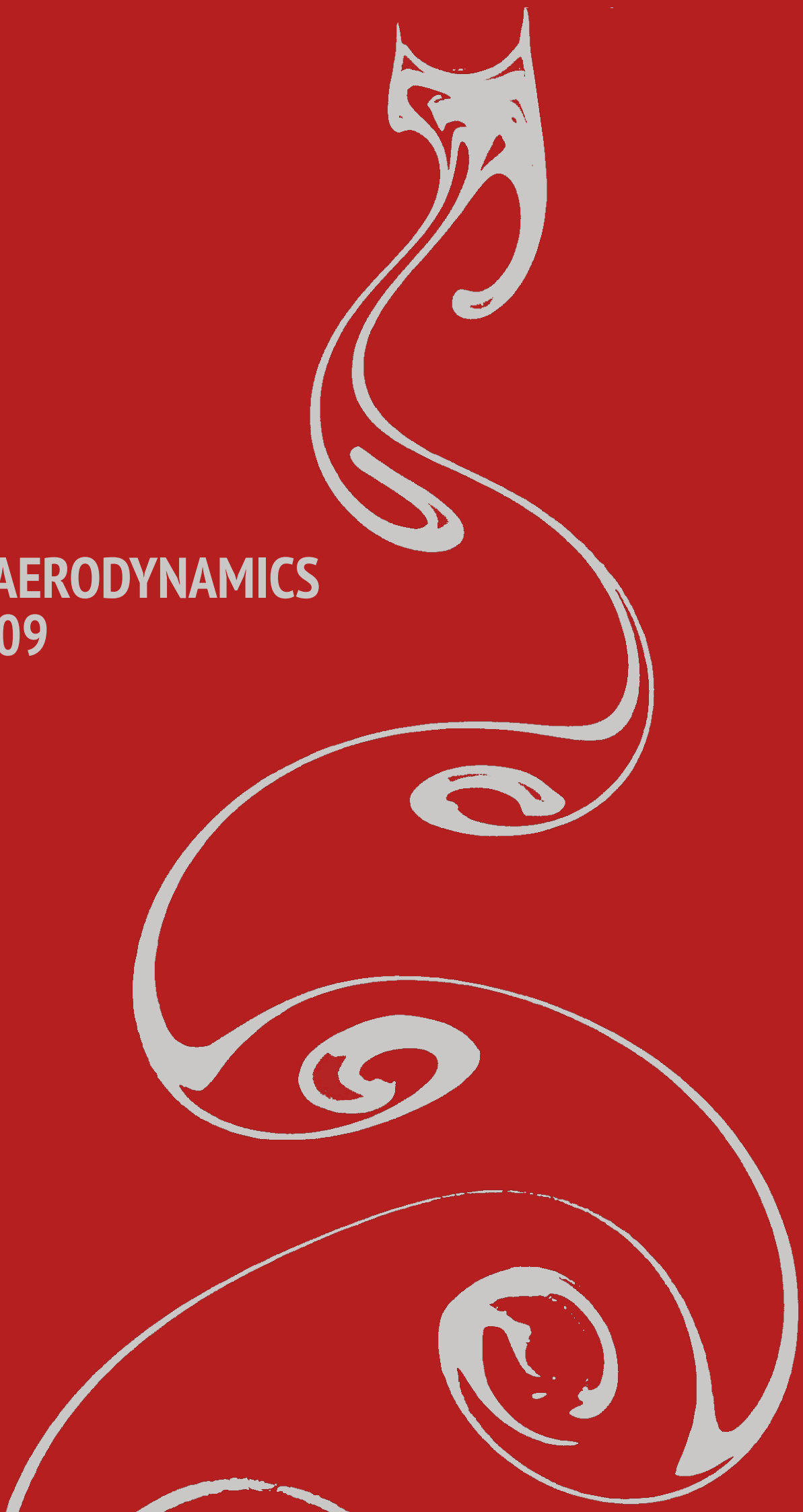


ME-445 AERODYNAMICS
Exercise 09
Week 9



Formula sheet

Cylindrical coordinates

$$\nabla \vec{u} = \left(\frac{\partial v_r}{\partial r}, \frac{1}{r} \frac{\partial v_\theta}{\partial \theta}, 0 \right)$$

$$\nabla \cdot \vec{u} = \frac{1}{r} \frac{\partial(rv_r)}{\partial r} + \frac{1}{r} \frac{\partial v_\theta}{\partial \theta}$$

$$\nabla \times \vec{u} = \left(0, 0, \frac{1}{r} \left[\frac{\partial(rv_\theta)}{\partial r} - \frac{\partial v_r}{\partial \theta} \right] \right)$$

Potential flow

$$v_r = \frac{\partial \phi}{\partial r} = \frac{1}{r} \frac{\partial \psi}{\partial \theta}, \quad v_\theta = \frac{1}{r} \frac{\partial \phi}{\partial \theta} = -\frac{\partial \psi}{\partial r}$$

Uniform parallel flow $w = \phi + i\psi = U_\infty e^{-i\alpha} z$

Potential vortex in z_0 $w = -\frac{i\gamma}{2\pi} \ln(z - z_0)$

Point source or sink in z_0 $w = \frac{Q}{2\pi} \ln(z - z_0)$

Source-sink doublet in z_0 $w = \frac{\mu}{2\pi(z - z_0)}$

$$\frac{dw}{dz} = u - iv$$

Milne-Thomson circle theorem:

$$g(z) = w(z) + \overline{w\left(\frac{a^2}{\bar{z}}\right)}$$

Thin airfoil theory

For a camber line with:

$$\frac{dy_c}{dx} = A_0 + \sum_{n=1}^{\infty} A_n \cos n\theta$$

$$\frac{x}{c} = \frac{(1 - \cos \theta)}{2}$$

we know:

$$k = 2U_\infty \left[(\alpha - A_0) \frac{\cos \theta + 1}{\sin \theta} + \sum_{n=1}^{\infty} A_n \sin n\theta \right]$$

$$A_0 = \frac{1}{\pi} \int_0^\pi \frac{dy_c}{dx} d\theta$$

$$A_n = \frac{2}{\pi} \int_0^\pi \frac{dy_c}{dx} \cos n\theta d\theta$$

$$C_l = 2\pi\alpha + \pi(A_1 - 2A_0)$$

$$C_{m,1/4} = -\frac{\pi}{4}(A_1 - A_2)$$

$$x_{cp} = \frac{1}{4} + \frac{\pi}{4C_l}(A_1 - A_2)$$

Finite wings with $AR=b^2/S$

Sign convention:

if induced velocity points downward: $w(y) > 0$, $\alpha_i(y) > 0$

if induced velocity points upward: $w < 0$, $\alpha_i < 0$

Prandtl's lifting-line theory

$$U_\infty \alpha_i(y_0) = w(y_0) = -\frac{1}{4\pi} \int_{-b/2}^{b/2} \frac{(d\Gamma/dy)}{y - y_0} dy$$

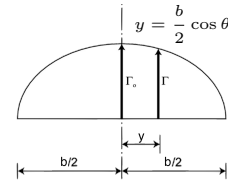
$$\alpha(y_0) = \alpha_{eff}(y_0) + \alpha_i(y_0)$$

Elliptical loading $\Gamma(y) = \Gamma_0 \sqrt{1 - \left(\frac{2y}{b}\right)^2}$

$$w = \frac{\Gamma_0}{2b}$$

$$\alpha_i = \frac{C_L}{\pi AR}$$

$$C_{D,i} = \frac{C_L^2}{\pi AR}$$



General loading $\Gamma(\theta) = 2bU_\infty \sum_{n=1}^{\infty} A_n \sin n\theta$

$$w(\theta) = U_\infty \sum_{n=1}^{\infty} n A_n \frac{\sin n\theta}{\sin \theta}$$

$$C_L = \pi A_1 AR$$

$$C_{D,i} = \frac{C_L^2}{\pi AR} (1 + \delta) \text{ with } \delta = \sum_{n=2}^{\infty} n (A_n/A_1)^2$$

Boundary Layer

Flat plate **laminar** boundary layer

$$\frac{\delta}{x} = \frac{5}{\sqrt{Re_x}} \quad \text{boundary layer growth}$$

$$C_f = \frac{1.328}{\sqrt{Re_x}} \quad \text{skin friction drag coefficient}$$

Flat plate **turbulent** boundary layer

$$\frac{\delta}{x} = \frac{0.37}{Re_x^{1/5}} \quad \text{boundary layer growth}$$

$$C_f = \frac{0.074}{Re_x^{1/5}} \quad \text{skin friction drag coefficient}$$

Miscellaneous

θ	0°	30°	45°	60°	90°
$\sin \theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos \theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0

water

kinematic viscosity $\nu = 1 \times 10^{-6} \text{ m}^2 \text{ s}^{-1}$

density $\rho = 1000 \text{ kg m}^{-3}$

air

kinematic viscosity $\nu = 1.5 \times 10^{-5} \text{ m}^2 \text{ s}^{-1}$

density $\rho = 1.2 \text{ kg m}^{-3}$

$$\sin (x \pm y)=\sin x \cos y \pm \cos x \sin y$$

$$\cos (x \pm y)=\cos x \cos y \mp \sin x \sin y$$

$$\cos 2 \theta=2 \cos ^2 \theta-1$$

$$\sin 2 \theta=2 \sin \theta \cos \theta$$

$$\sin 3 \theta=3 \sin \theta-4 \sin ^3 \theta$$

$$\cos 3 \theta=4 \cos ^3 \theta-3 \cos \theta$$

$$\int_0^{\pi} \cos \theta \mathrm{d} \theta=0$$

$$\int_0^{\pi} \sin \theta \mathrm{d} \theta=2$$

$$\int_0^{\pi} \cos ^2 \theta \mathrm{d} \theta=\int_0^{\pi} \sin ^2 \theta \mathrm{d} \theta=\frac{\pi}{2}$$

$$\int_0^{\pi} \frac{\cos n \theta}{\cos \theta-\cos \theta_1} \mathrm{d} \theta=\pi \frac{\sin n \theta_1}{\sin \theta_1} \quad n=0,1,2, \ldots$$

$$\int_0^{\pi} \frac{\sin n \theta \sin \theta}{\cos \theta-\cos \theta_1} \mathrm{d} \theta=-\pi \cos n \theta_1 \quad n=1,2,3, \ldots$$

1. The circulation around the wing at any point y is denoted Γ . If the circulation has a parabolic form:

$$\Gamma = \Gamma_0 \left[1 - \left(\frac{2y}{b} \right)^2 \right]$$

- (a) What is the downward induced velocity behind the wing with the span b ?

Solution:

$$\begin{aligned} \Gamma &= \Gamma_0 \left[1 - \left(\frac{2y}{b} \right)^2 \right] = \Gamma_0 - \Gamma_0 \frac{4y^2}{b^2} \\ \frac{d\Gamma}{dy} &= -\frac{8\Gamma_0 y}{b^2} \\ w(y_0) &= -\frac{1}{4\pi} \int_{-b/2}^{b/2} \frac{d\Gamma/dy}{y - y_0} dy = \frac{2\Gamma_0}{\pi b^2} \int_{-b/2}^{b/2} \frac{y}{y - y_0} dy \end{aligned}$$

At mid-span:

$$w(y_0 = 0) = \frac{2\Gamma_0}{\pi b}$$

- (b) Compare w at mid-span with that obtained when the same lift is distributed elliptically.

Solution: Downwash at mid-span:

$$w_q(y_0 = 0) = \frac{2\Gamma_0}{\pi b}$$

For the elliptical distribution:

$$w_e(y_0 = 0) = \frac{\Gamma_0}{2b}$$

Lift force:

$$L_q = \rho U_\infty \int_{-b/2}^{b/2} \Gamma_0 \left(1 - \frac{4y^2}{b^2} \right) dy = \frac{2}{3} \rho U_\infty \Gamma_0 b$$

For the elliptical distribution:

$$L_e = \frac{\pi}{4} \rho U_\infty \Gamma_0 b$$

For the same lift:

$$\begin{aligned} \frac{\pi}{4} \Gamma_{0,e} &= \frac{2}{3} \Gamma_{0,q} \\ \frac{w_{0,q}}{w_{0,e}} &= \frac{2\Gamma_{0,q}}{\pi b} \frac{2b}{\Gamma_{0,e}} = \frac{4}{\pi} \frac{w_{0,q}}{w_{0,e}} = \frac{3}{2} \end{aligned}$$

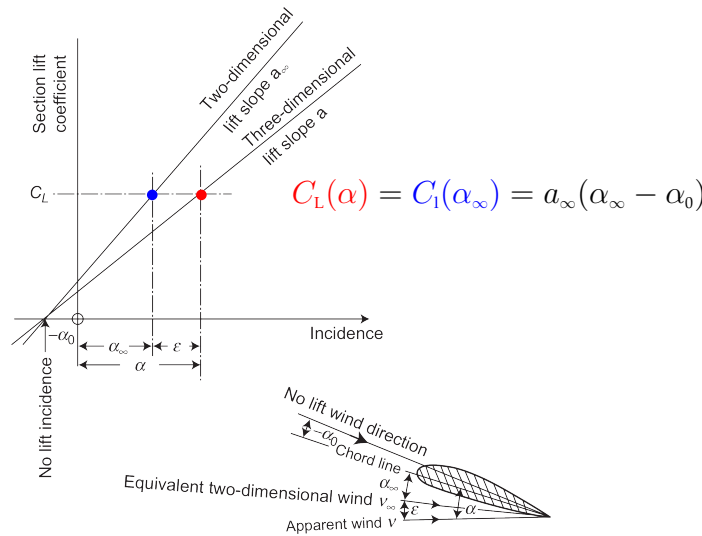
→ The downwash of the elliptic wing is 2/3 of the downwash of the quadratic wing.

2. Consider a rectangular wing with an aspect ratio $AR = 6$, an induced drag factor $\delta_{AR=6} = 0.055$, and a zero-lift angle of attack $\alpha_0 = -2^\circ$. At an angle of attack of 3.4° , the induced drag coefficient for this wing is $C_{D,i,AR=6} = 0.01$. Calculate the induced drag coefficient for another rectangular wing with the same airfoil section at the same angle of attack, but with an aspect ratio $AR = 10$ with $\delta_{AR=10} = 0.105$.

(a) Show that the lift coefficient slope for the 3D wing can be expressed as:

$$a = \frac{dC_L}{d\alpha} = \frac{a_\infty}{1 + \frac{a_\infty}{\pi AR}(1 + \delta)}$$

with a_∞ the lift coefficient slope for the 2D airfoil



Solution: From the lift coefficient distribution

$$C_L(\alpha) = C_L(\alpha_\infty) = a_\infty(\alpha_\infty - \alpha_0)$$

$$C_L = a_\infty(\alpha - \alpha_i - \alpha_0)$$

with

$$\alpha_i = \frac{C_{D,i}}{C_L} = \frac{C_L}{\pi AR}(1 + \delta)$$

$$C_L = a_\infty\alpha - a_\infty \frac{C_L}{\pi AR}(1 + \delta) - a_\infty\alpha_0$$

$$C_L \left(1 + \frac{a_\infty}{\pi AR}(1 + \delta) \right) = a_\infty\alpha - a_\infty\alpha_0$$

$$\Rightarrow a = \frac{dC_L}{d\alpha} = \frac{\partial C_L}{\partial \alpha} = \frac{a_\infty}{1 + \frac{a_\infty}{\pi AR}(1 + \delta)}$$

- (b) Calculate the induced drag coefficient for another rectangular wing with the same airfoil section at the same angle of attack, but with an aspect ratio $AR = 10$ with $\delta_{AR=10} = 0.105$.

Solution: For $AR = 6$:

$$C_L^2 = \frac{\pi AR C_{D,i}}{1 + \delta} = \frac{\pi \times 6 \times 0.01}{1.055} = 0.1787$$

$$C_L = 0.423$$

$$\frac{dC_L}{d\alpha} = \frac{\Delta C_L}{\Delta \alpha} = \frac{0.423}{3.4 + 2} = 0.078 \text{ deg}^{-1} = 4.485^{-1}$$

$$\frac{dC_L}{d\alpha} = a = \frac{a_\infty}{1 + \frac{a_\infty}{\pi AR}(1 + \delta)}$$

$$a_\infty = \frac{a_{AR=6}}{1 - \frac{a_{AR=6}}{\pi AR}(1 + \delta_{AR=6})} = \frac{4.485}{1 - \frac{4.485}{\pi \times 6} \times 1.055} = 5.99^{-1}$$

The second wing with $AR = 10$ has the same 2D airfoil profiles, thus the lift coefficient slopes are as follows:

$$a_{AR=10} = \frac{a_\infty}{1 + \frac{a_\infty}{\pi AR}(1 + \delta_{AR=10})} = \frac{5.99^{-1}}{1 + \frac{5.99^{-1}}{\pi \times 10} \times 1.105} = 4.95^{-1} = 0.0864 \text{ deg}^{-1}$$

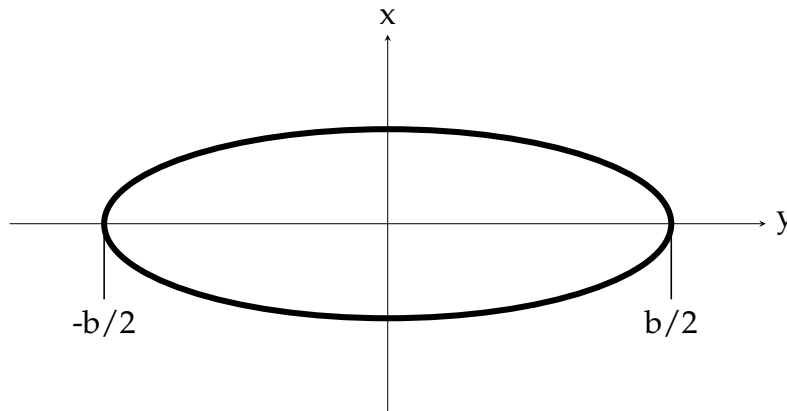
$$C_L = a_{AR=10}(\alpha - \alpha_0) = 0.0864 \times (3.4 + 2) = 0.467$$

$$C_{D,i} = \frac{C_L^2}{\pi AR}(1 + \delta) = \frac{0.467^2}{\pi \times 10} \times 1.105 = 0.0077$$

3. Consider an untwisted, unswept wing which has an elliptical lift distribution $\Gamma_{\text{elliptical}}$ under normal flight conditions. The wing is equipped with a system to reduce torsional moments when encountering gusts. When approaching a storm front, rudder and flap motions alter the span-wise circulations distribution to $\Gamma_{\text{altered}}(\theta) = \Gamma_0 \left[\sin \theta - \frac{1}{4} \sin 3\theta \right]$ with Γ_0 the same constant as in the expression for $\Gamma_{\text{elliptical}}$.

(a) Draw a sketch of the most probable planform area of this wing seen from above.

Solution: Elliptical planform no sweep



(b) Determine the span-wise lift distribution under normal conditions $L_{\text{elliptical}}(\theta)$.

Solution:

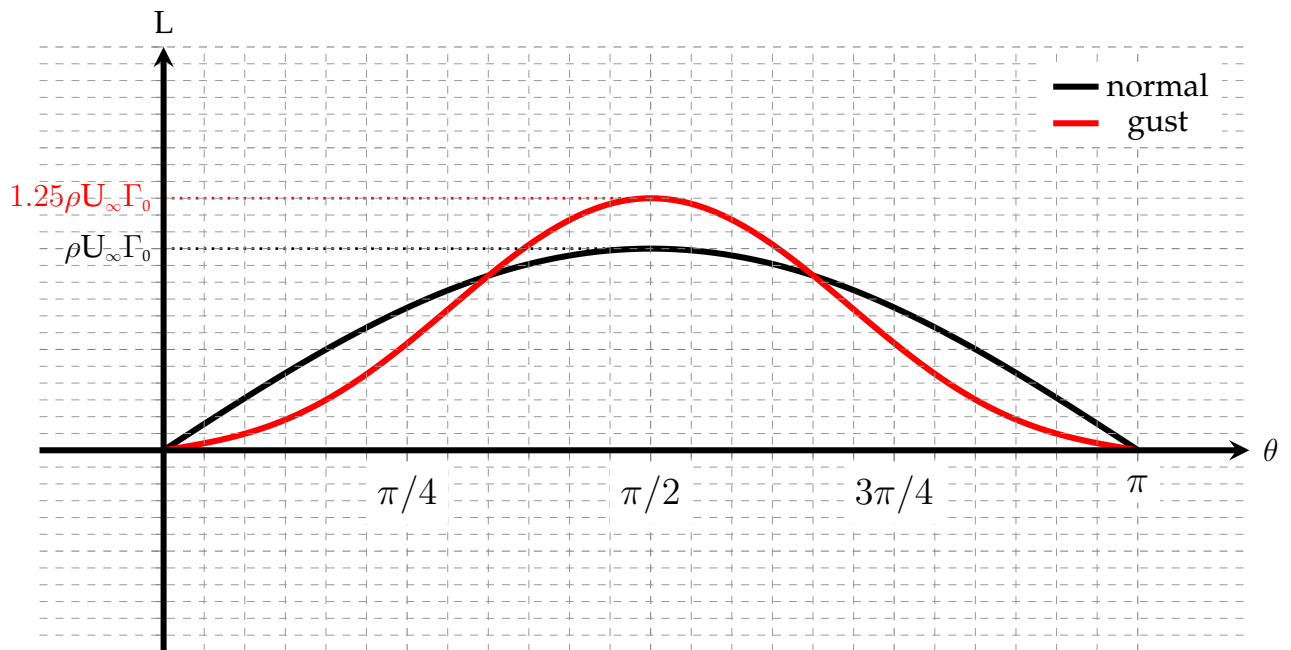
$$\begin{aligned} L_{\text{normal}}(\theta) &= \rho U_{\infty} \Gamma_0 \sqrt{1 - \left(\frac{2y}{b} \right)^2} \\ &= \rho U_{\infty} \Gamma_0 \sin \theta \end{aligned}$$

(c) Determine the span-wise lift distribution under altered gusty conditions $L_{\text{altered}}(\theta)$.

Solution:

$$L_{\text{altered}}(\theta) = \rho U_{\infty} \Gamma_0 \left[\sin \theta - \frac{1}{4} \sin 3\theta \right]$$

(d) Draw both lift distributions in function of the span-wise location and indicate explicitly the extrema values.



(e) Determine for both flight conditions the total lift coefficient C_L

Solution:
$$\int_{-b/2}^{b/2} \Gamma(y) dy = \frac{b}{2} \int_0^\theta \Gamma(\theta) \sin \theta d\theta$$

$$C_L = \frac{L}{\frac{1}{2} \rho U_\infty^2 S} = \frac{\rho U_\infty}{\frac{1}{2} \rho U_\infty^2 S} \int_{-b/2}^{b/2} \Gamma(y) dy = \frac{b}{U_\infty S} \int_0^\theta \Gamma(\theta) \sin \theta d\theta$$

Elliptical distribution:

$$\begin{aligned} C_L &= \frac{b}{U_\infty S} \int_0^\theta \Gamma_0 \sin^2 \theta d\theta \\ &= \frac{b \pi \Gamma_0}{S 2 U_\infty} \end{aligned}$$

Altered distribution:

$$\begin{aligned} C_L &= \frac{b}{U_\infty S} \int_0^\theta \Gamma_0 \sin^2 \theta d\theta + \underbrace{\frac{b}{U_\infty S} \int_0^\theta -\frac{\Gamma_0}{4} \sin 3\theta \sin \theta d\theta}_{=0} \\ &= \frac{b \pi \Gamma_0}{S 2 U_\infty} \end{aligned}$$

Solution:

- (f) Determine the extra propulsion force that is required to remain the same flight altitude when the lift distribution is altered from $L_{\text{elliptical}}(\theta)$ to $L_{\text{altered}}(\theta)$.

(Hint: use $C_{D,i} = \frac{C_L^2}{\pi AR} (1 + \delta)$ with $\delta = \sum_{n=2}^{\infty} n (A_n/A_1)^2$)

Solution:

General loading: $\Gamma(\theta) = 2bU_{\infty} \sum_{n=1}^{\infty} A_n \sin n\theta$ and $\delta = \sum_{n=2}^{\infty} n (A_n/A_1)^2$

Elliptical distribution:

$$\Gamma(\theta) = \Gamma_0 \sin \theta = 2bU_{\infty} A_1 \sin \theta \Rightarrow A_1 = \frac{\Gamma_0}{2bU_{\infty}}$$

$$\delta = 0$$

Altered distribution:

$$\Gamma(\theta) = \Gamma_0 \left[\sin \theta - \frac{1}{4} \sin 3\theta \right] = 2bU_{\infty} A_1 \sin \theta + = 2bU_{\infty} A_3 \sin 3\theta$$

$$\Rightarrow A_1 = \frac{\Gamma_0}{2bU_{\infty}} \quad , \quad A_3 = -\frac{\Gamma_0}{8bU_{\infty}}$$

$$\delta = 3 \left(\frac{A_3}{A_1} \right)^2 = 3 \left(\frac{2}{-8} \right)^2 = \frac{3}{16}$$

Extra propulsion necessary to compensate the increased drag

$$\begin{aligned} \Delta C_{D,i} &= \delta_{\text{altered}} \frac{C_L^2}{\pi AR} \\ &= \frac{3}{16} \frac{C_L^2}{\pi AR} = \frac{3}{16} \frac{b^2 \pi^2 \Gamma_0^2}{S^2 4U_{\infty}^2} \frac{S}{\pi b^2} = \frac{3}{128} \frac{\pi \Gamma_0^2}{1/2 U_{\infty}^2 S} \\ \Rightarrow \Delta D &= \frac{3}{128} \rho \pi \Gamma_0^2 \end{aligned}$$

- (g) Can you think of a reason why it would help to change the lift distribution in the proposed way in terms of gust load alleviation?

Solution: Higher load distribution near the centre reduces the bending moments near the tips.