Fall 2024

ME-445 AERODYNAMICS

03 - Potential flow theory

K. Mulleners

Potential flow theory

Potential flow background

Flow potential and stream function

Elementary potential flows

Combination of elementary potential flows

Magnus effect

Conformal mapping

Joukowski theory

Potential flow

- when is flow a potential flow? ☐
- definition of flow potential ☐
- definition of stream function ☐
- Laplace equations ☐
- Cauchy-Riemann equations ☐
- circulation and vorticity ☐

FAQ

Is a potential flow inviscid?

A potential flow is irrotational & incompressible

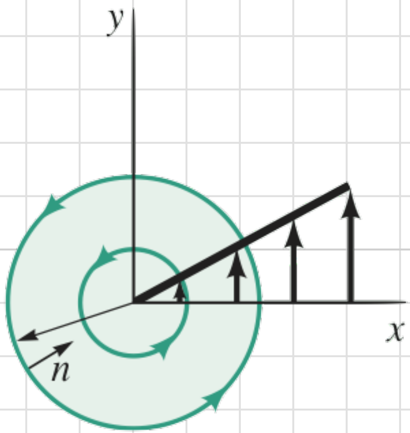
FAQ

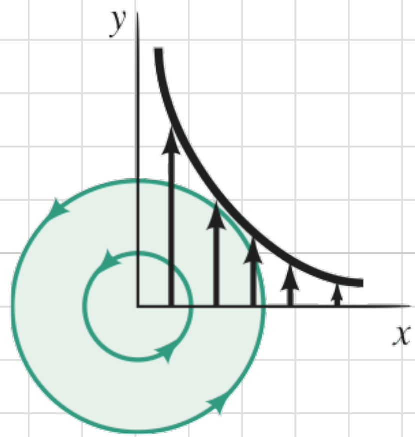
$$\nabla \cdot \vec{u} = \frac{1}{r}\frac{\partial (r v_r)}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta}$$

$$\nabla \times \vec{u} = \left(0, 0, \frac{1}{r}\left[\frac{\partial (r v_\theta)}{\partial r} - \frac{\partial v_r}{\partial \theta}\right]\right)$$

Can there be vortices in a potential flow?

Solid body rotation vs free vortex





Potential flow

Potential flow

Potential flow

Potential flow

Complex potential and it's properties

definition of complex potential ☐

$$\bar{w} = ?, \quad \frac{\mathrm{d}w}{\mathrm{d}z} = ?, \quad \frac{\mathrm{d}w}{\mathrm{d}z} = ?, \quad \frac{\mathrm{d}\bar{w}}{\mathrm{d}z} = ?, \quad \left|\frac{\mathrm{d}w}{\mathrm{d}z}\right| = ?$$

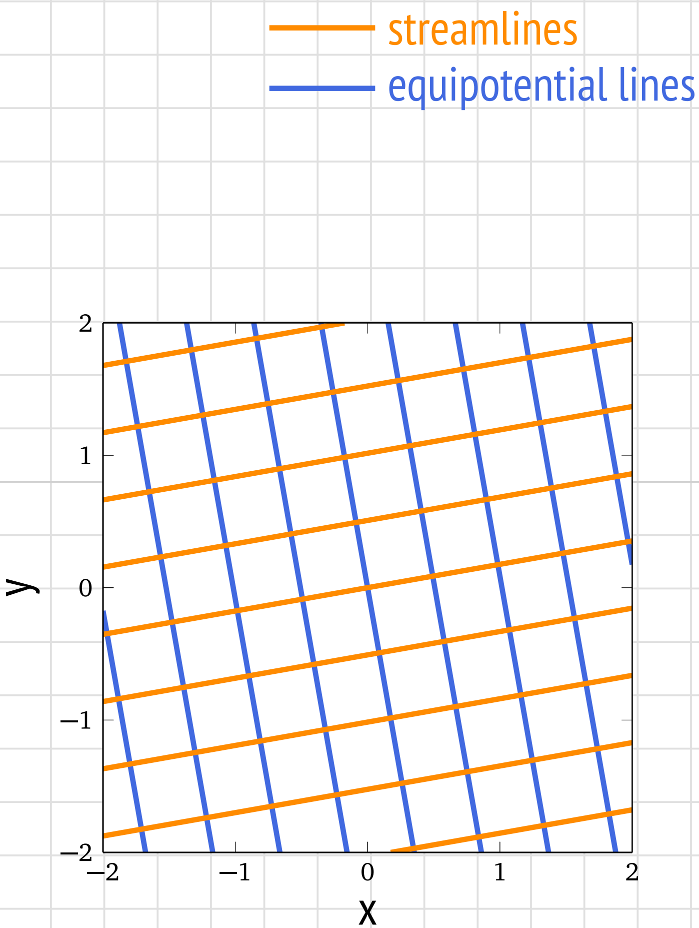
$$w + \bar{w} = ?, \quad w - \bar{w} = ?,$$

Elementary potential flows

Uniform parallel flow

Elementary potential flows

Uniform parallel flow



Elementary potential flows

Potential vortex

In 2D:

$$v_r = \frac{\partial \phi}{\partial r} = \frac{1}{r}\frac{\partial \psi}{\partial \theta}$$

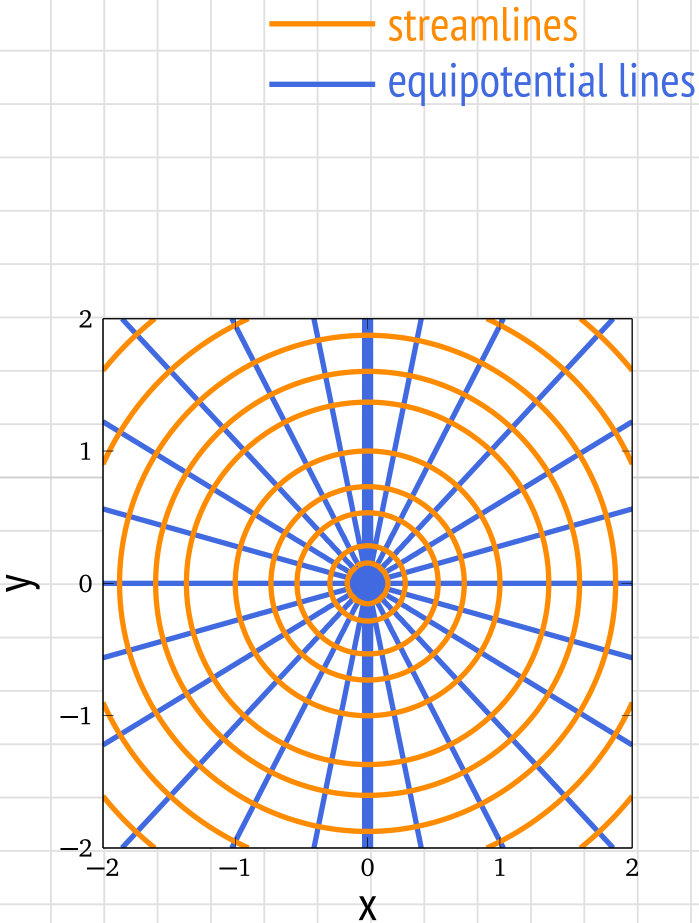
$$v_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = -\frac{\partial \psi}{\partial r}$$

$$\nabla \cdot \vec{u} = \frac{1}{r}\frac{\partial (r v_r)}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta}$$

$$\nabla \times \vec{u} = \left(0, 0, \frac{1}{r}\left[\frac{\partial (r v_\theta)}{\partial r} - \frac{\partial v_r}{\partial \theta}\right]\right)$$

Elementary potential flows

Potential vortex

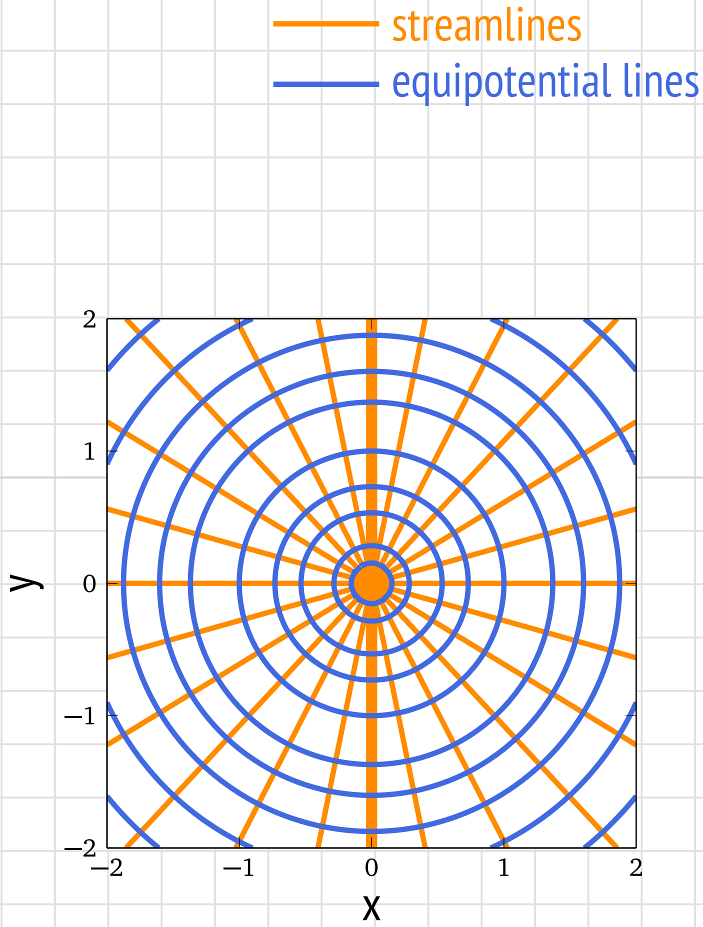


Elementary potential flows

Point source or sink

Elementary potential flows

Point source or sink

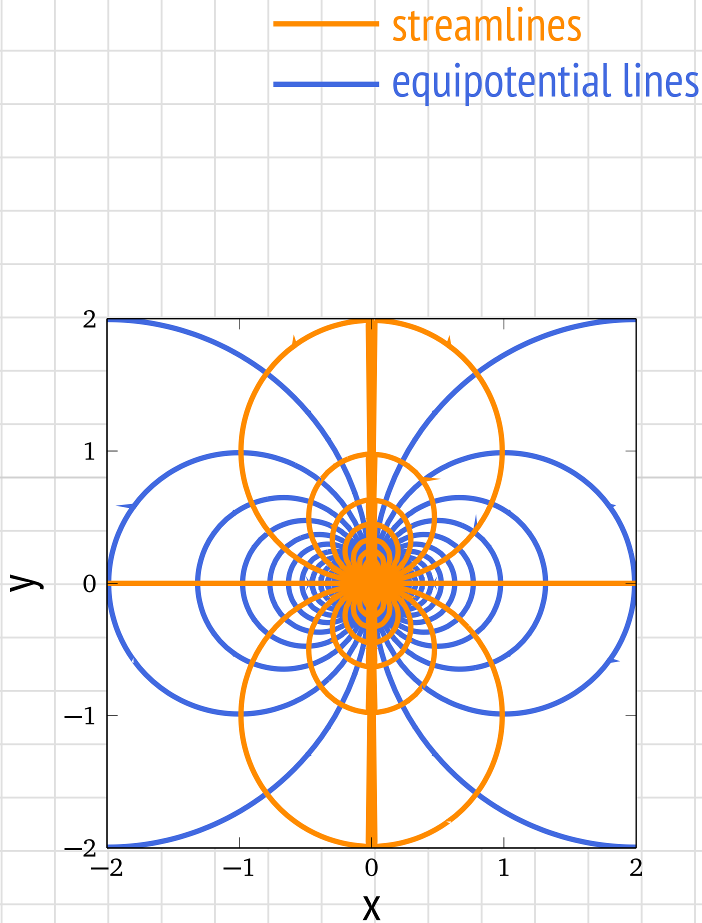


Elementary potential flows

Source-sink doublet

Elementary potential flows

Source-sink doublet

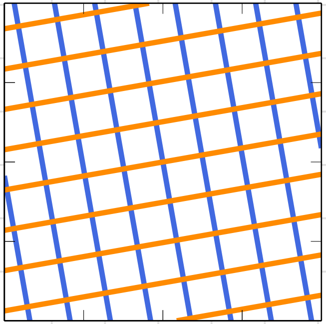
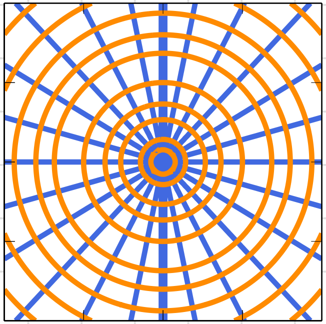
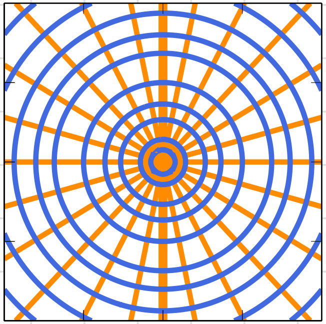
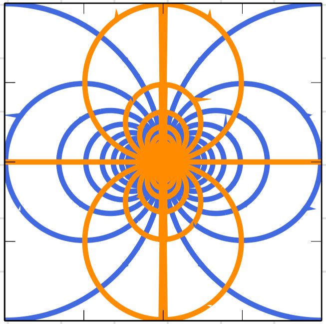


Elementary potential flows

streamlines
equipotential lines

Summary

	w	ϕ	ψ
a. Uniform parallel flow	$U_\infty e^{-i\alpha} z$	$U_\infty (x\cos\alpha + y\sin\alpha)$	$U_\infty (y\cos\alpha - x\sin\alpha)$
b. Potential vortex	$-\frac{i\gamma}{2\pi}\ln z$	$\frac{\gamma}{2\pi}\theta$	$-\frac{\gamma}{2\pi}\ln r$
c. Point source or sink	$\frac{Q}{2\pi}\ln z$	$\frac{Q}{2\pi}\ln r$	$\frac{Q}{2\pi}\theta$
d. Source-sink doublet	$\frac{\mu}{2\pi z} e^{i\alpha}$	$\frac{\mu}{2\pi r}\cos(\theta-\alpha)$	$-\frac{\mu}{2\pi r}\sin(\theta-\alpha)$

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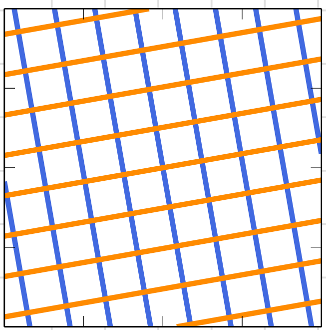
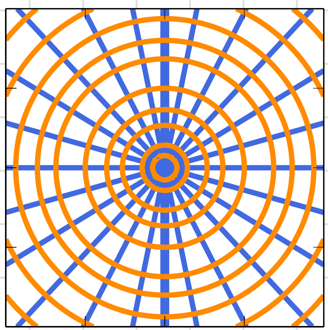
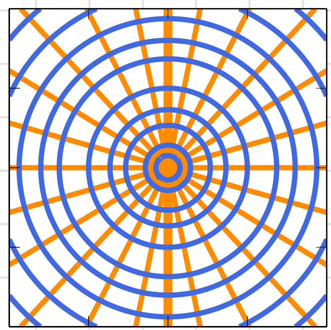
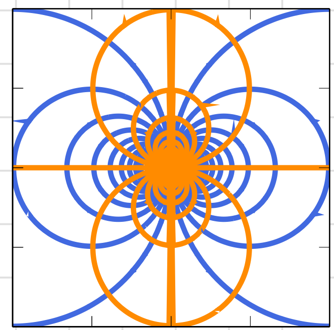
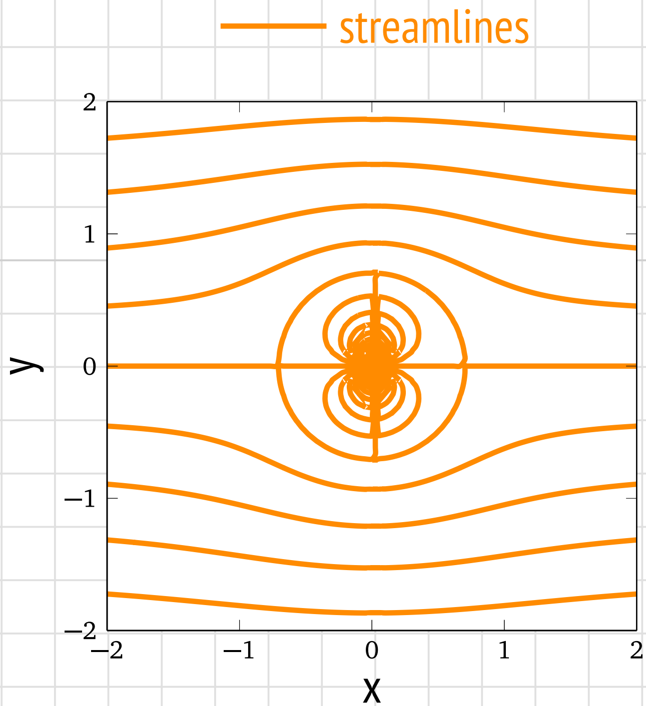





Combination of elementary potential flows

Source-sink dipole in free stream → flow around cylinder



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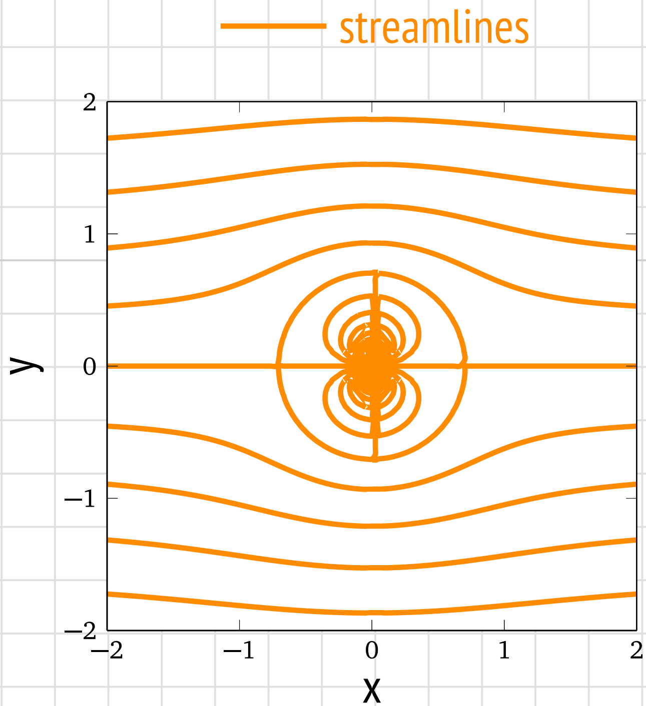


Combination of elementary potential flows

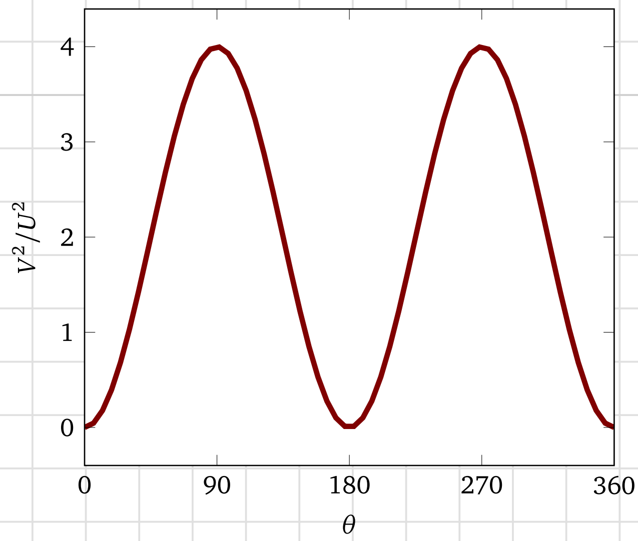
Milne-Thomson circle theorem

Consider a flow field represented by the complex potential $w(z)$. If a circle $|z| = a$ is placed into this flow field, the new complex potential $g(z)$ of this flow field equals:

$$g(z) = w(z) + \overline{w\left(\frac{a^2}{\bar{z}}\right)}$$

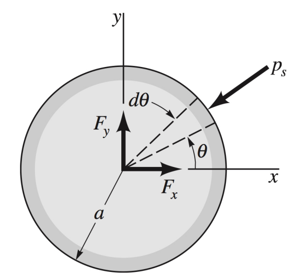
Combination of elementary potential flows

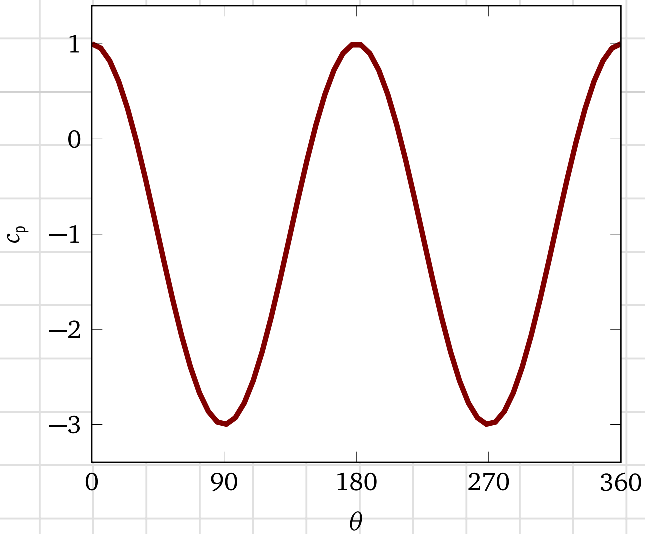
Flow around cylinder



Combination of elementary potential flows

Flow around cylinder





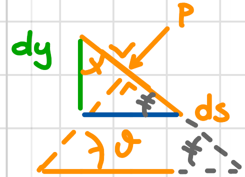
Combination of elementary potential flows

Flow around cylinder

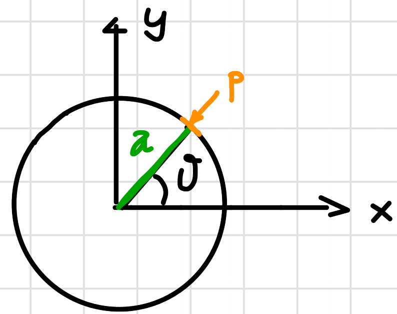
N = normal force due to pressure distribution

$$= -\int_{\text{upper}} p \underbrace{\sin\vartheta\, a\, d\vartheta}_{dx} + \int_{\text{lower}} p \underbrace{\sin\vartheta\, a\, d\vartheta}_{dx}$$

positive p on upper surface yields negative contribution to normal force



$dx = \sin\vartheta\, ds$
$dy = \cos\vartheta\, ds$
$ds = a\, d\vartheta$



upper surface $\rightarrow \vartheta = 0 : \pi$
lower surface $\rightarrow \vartheta = \pi : 2\pi$

$$= -\int_0^{\pi} p \sin\vartheta\, a\, d\vartheta + \int_{2\pi}^{\pi} p \sin\vartheta\, a\, d\vartheta$$

$$= -\int_0^{\pi} p \sin\vartheta\, a\, d\vartheta - \int_{\pi}^{2\pi} p \sin\vartheta\, a\, d\vartheta$$

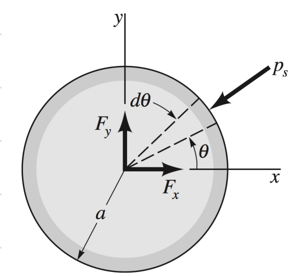
$$\frac{N}{\frac{1}{2}\rho U^2 a} = -\int_0^{2\pi} \frac{p}{\frac{1}{2}\rho U^2} \sin\vartheta \frac{a}{a}\, d\vartheta + \underbrace{\int_0^{2\pi} \frac{p_\infty}{1/2\,\rho U^2} \sin\vartheta\, d\vartheta}_{=0}$$

surface pressure distribution ☐

lift ☐

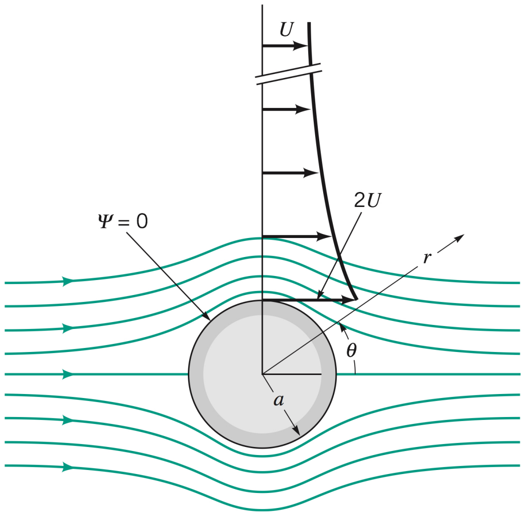
Combination of elementary potential flows

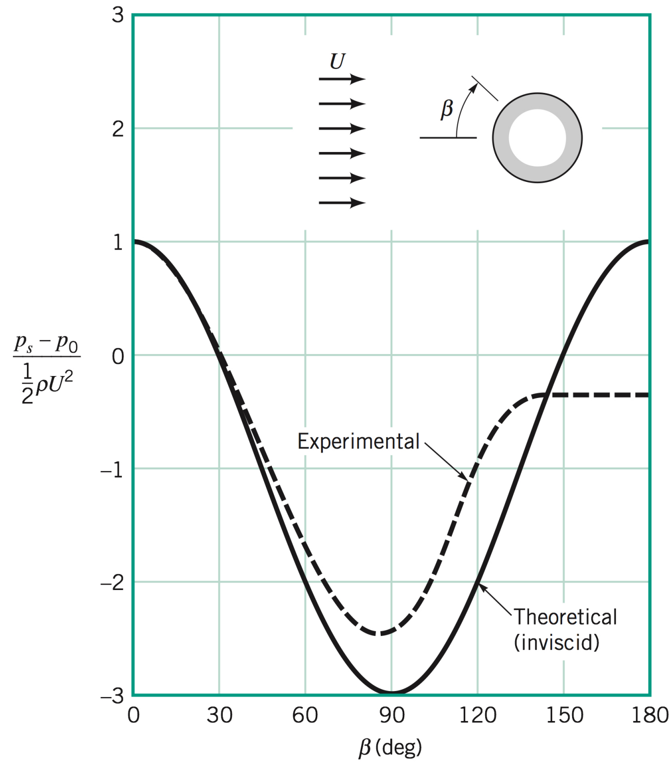
Flow around cylinder



Combination of elementary potential flows

Flow around cylinder



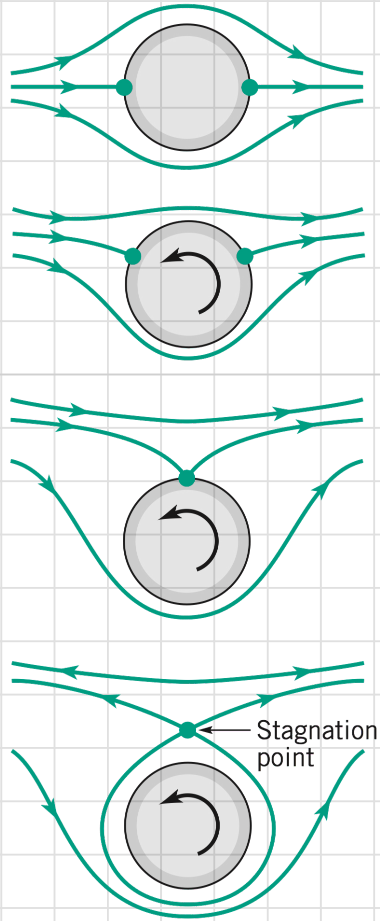


Combination of elementary potential flows

Flow around cylinder with lift

Combination of elementary potential flows

Flow around cylinder with lift

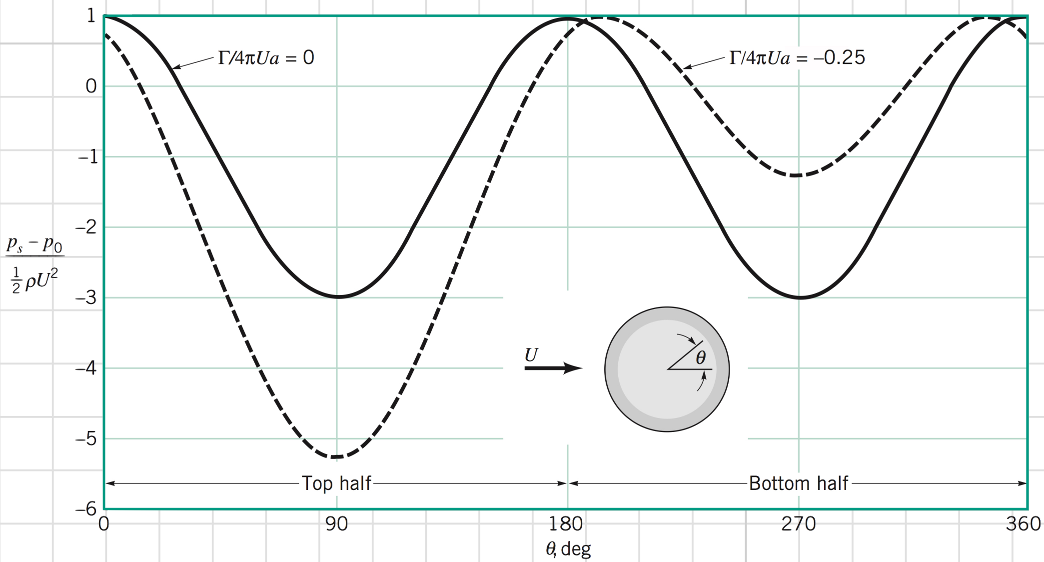


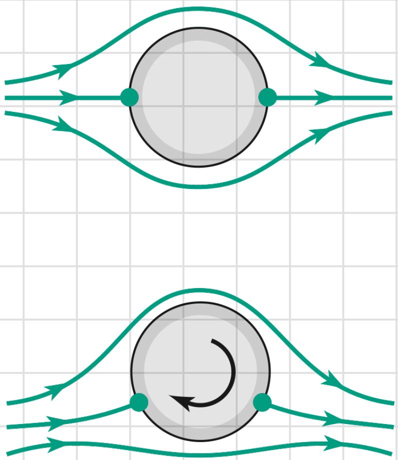
Combination of elementary potential flows

Flow around cylinder with lift

Combination of elementary potential flows

Flow around cylinder with lift





Combination of elementary potential flows

Flow around cylinder with lift

Combination of elementary potential flows

Flow around cylinder with lift

Flow around cylinder with lift

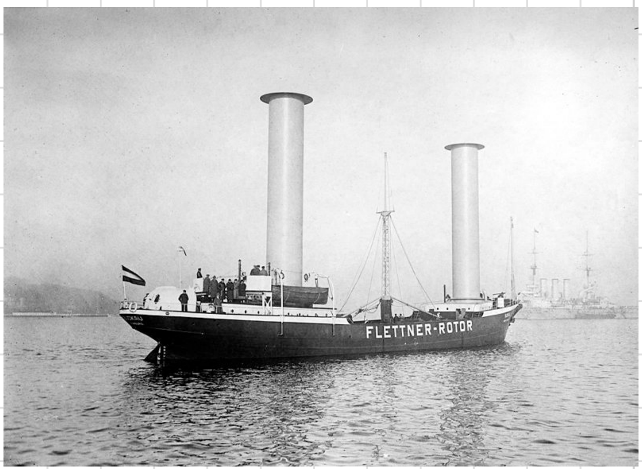
Magnus effect

DIE NATURWISSENSCHAFTEN

Dreizehnter Jahrgang 6. Februar 1925 Heft 6

Magnuseffekt und Windkraftschiff[1]).

Von L. PRANDTL, Göttingen.



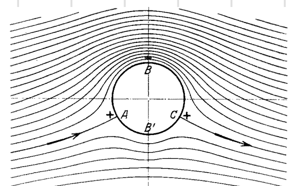


Fig. 6.

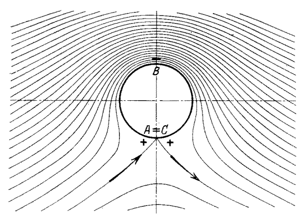


Fig. 7.

Fig. 6 u. 7. Zirkulationsströmungen, aus Fig. 3 u. 4 durch Überlagerung entstanden.

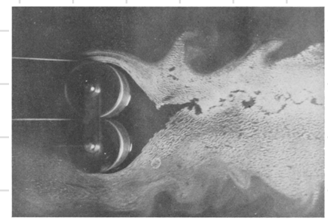


Fig. 16. Zwei gegenläufig rotierende Zylinder.

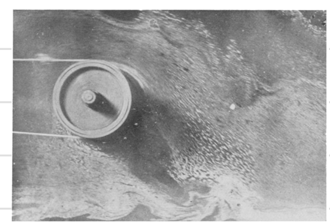
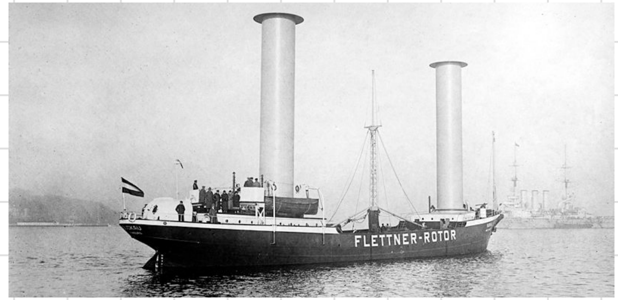


Fig. 17. Ein rotierender Zylinder.

Flow around cylinder with lift

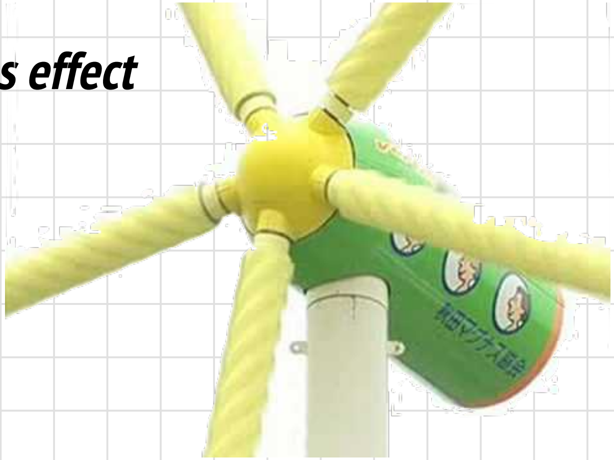
Magnus effect

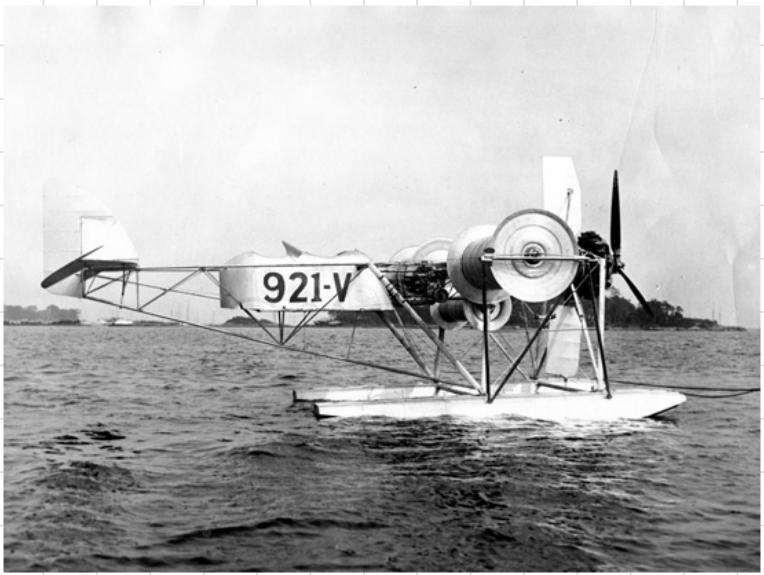


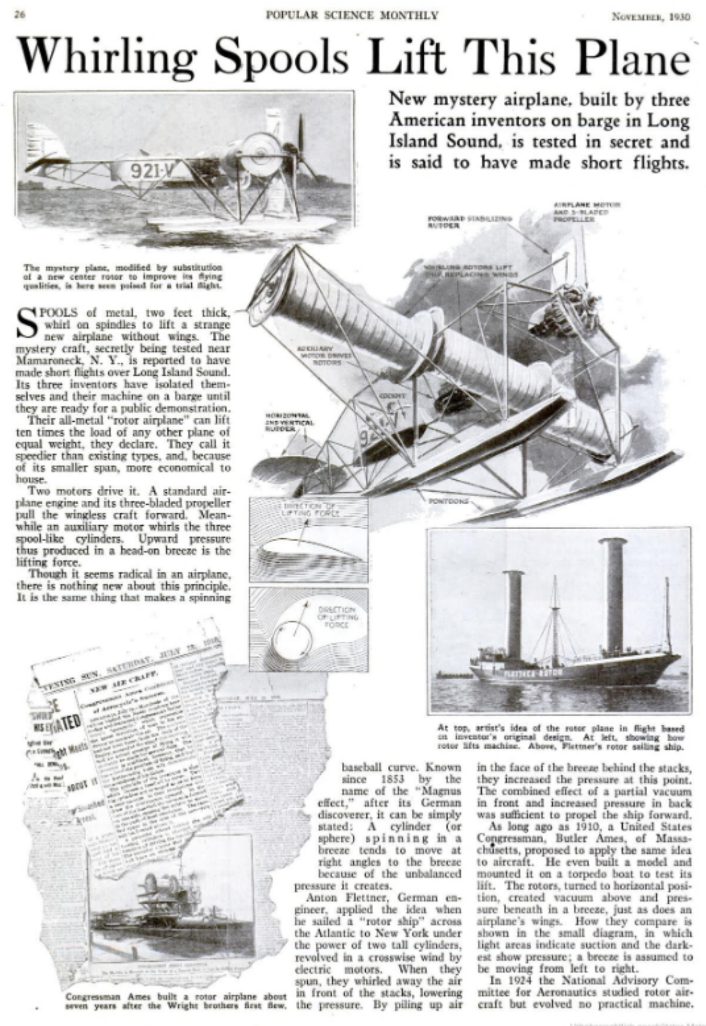
Flettner rotor with Thom-disks

Flow around cylinder with lift

Magnus effect







Whirling Spools Lift This Plane

New mystery airplane, built by three American inventors on barge in Long Island Sound, is tested in secret and is said to have made short flights.

The mystery plane, modified by substitution of a new center rotor to improve its flying qualities, is here seen poised for a trial flight.

At top, artist's idea of the rotor plane in flight based on inventor's original design. At left, showing how rotor lifts machine. Above, Flettner's rotor sailing ship.

Congressman Ames built a rotor airplane about seven years after the Wright brothers first flew.

SPOOLS of metal, two feet thick, whirl on spindles to lift a strange new airplane without wings. The mystery craft, secretly being tested near Mamaroneck, N. Y., is reported to have made short flights over Long Island Sound. Its three inventors have isolated themselves and their machine on a barge until they are ready for a public demonstration.

Their all-metal "rotor airplane" can lift ten times the load of any other plane of equal weight, they declare. They call it speedier than existing types, and, because of its smaller span, more economical to house.

Two motors drive it. A standard airplane engine and its three-bladed propeller pull the wingless craft forward. Meanwhile an auxiliary motor whirls the three spool-like cylinders. Upward pressure thus produced in a head-on breeze is the lifting force.

Though it seems radical in an airplane, there is nothing new about this principle. It is the same thing that makes a spinning baseball curve. Known since 1853 by the name of the "Magnus effect," after its German discoverer, it can be simply stated: A cylinder (or sphere) spinning in a breeze tends to move at right angles to the breeze because of the unbalanced pressure it creates.

Anton Flettner, German engineer, applied the idea when he sailed a "rotor ship" across the Atlantic to New York under the power of two tall cylinders, revolved in a crosswise wind by electric motors. When they spun, they whirled away the air in front of the stacks, lowering the pressure. By piling up air in the face of the breeze behind the stacks, they increased the pressure at this point. The combined effect of a partial vacuum in front and increased pressure in back was sufficient to propel the ship forward.

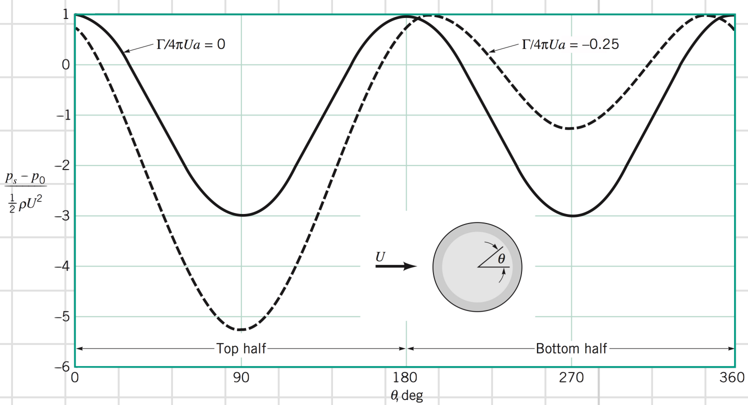
As long ago as 1910, a United States Congressman, Butler Ames, of Massachusetts, proposed to apply the same idea to aircraft. He even built a model and mounted it on a torpedo boat to test its lift. The rotors, turned to horizontal position, created vacuum above and pressure beneath in a breeze, just as does an airplane's wings. How they compare is shown in the small diagram, in which light areas indicate suction and the darkest show pressure; a breeze is assumed to be moving from left to right.

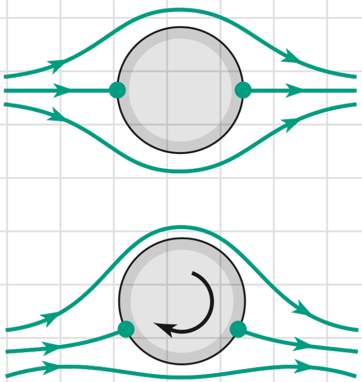
In 1924 the National Advisory Committee for Aeronautics studied rotor aircraft but evolved no practical machine.

Summary

with $a = \sqrt{\dfrac{\mu}{2\pi U_\infty}}$

	Source-sink doublet in free stream	Source-sink doublet in free stream + vortex
w	$U_\infty\left(\dfrac{a^2}{z}+z\right)$	$U_\infty\left(\dfrac{a^2}{z}+z\right) - \dfrac{i\Gamma}{2\pi}\ln z$
V_θ	$-2U_\infty \sin(\theta)$	$-2U_\infty \sin(\theta) + \dfrac{\Gamma}{2\pi a}$
C_p	$1 - 4\sin^2(\theta)$	$1-4\sin^2(\theta)-\left(\dfrac{\Gamma}{2\pi a U_\infty}\right)^2+2\sin(\theta)\dfrac{\Gamma}{\pi a U_\infty}$
L , D	0, 0	$-\rho U_\infty \Gamma$, 0





Conformal mapping

A conformal mapping function is an analytical function that preserves the local angle (but not necessarily lengths) and whose derivative is non-zero everywhere

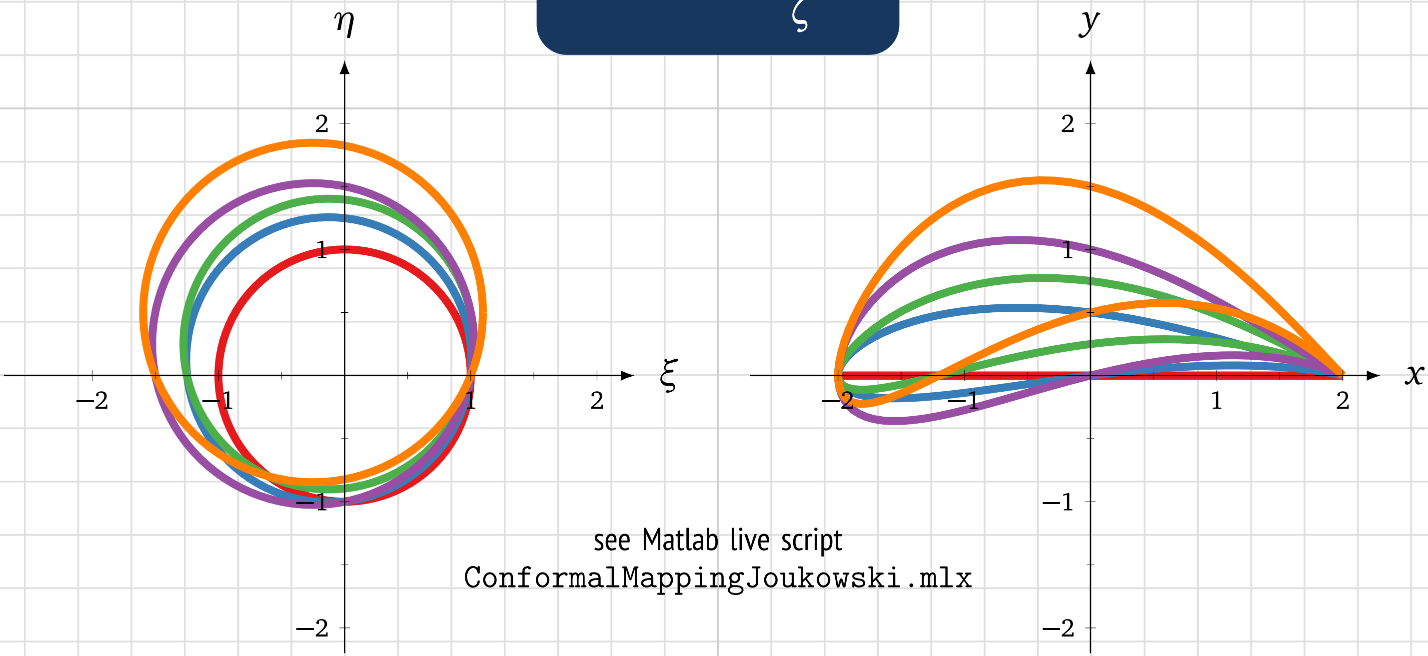
see Matlab live script `ConformalMappingExamples.mlx`
for examples of conformal mappings

Conformal mapping

A conformal mapping function is an analytical function that preserves the local angle (but not necessarily lengths) and whose derivative is non-zero everywhere

Joukowski transform:

$$z = \zeta + \frac{a^2}{\zeta}$$



see Matlab live script

`ConformalMappingJoukowski.mlx`

Joukowski airfoils

are airfoils that results from transforming a circle using a conformal mapping of the form:

$$z = \zeta + \frac{a^2}{\zeta}$$

$z = x + iy$ is the complex coordinate in the airfoil plane, $\zeta = \xi + i\eta$ is the complex coordinate in the circle plane.

The inverse transformation is: $$\zeta = \frac{1}{2}z \pm \left[\left(\frac{1}{2}z\right)^2 - a^2\right]^{1/2}$$

Joukowski airfoils

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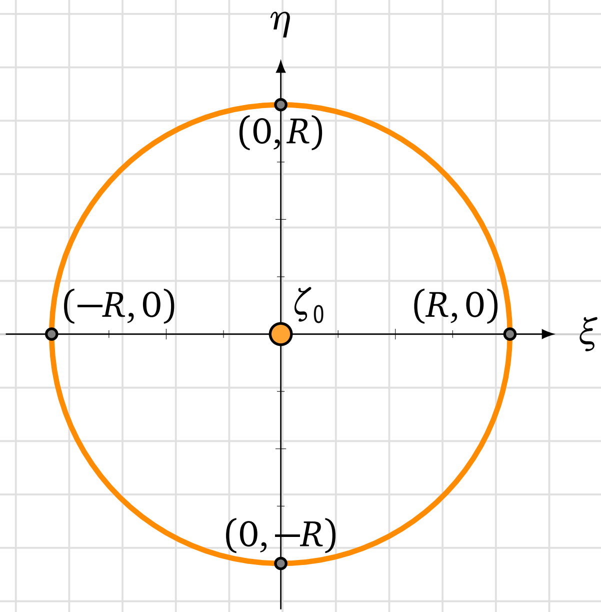
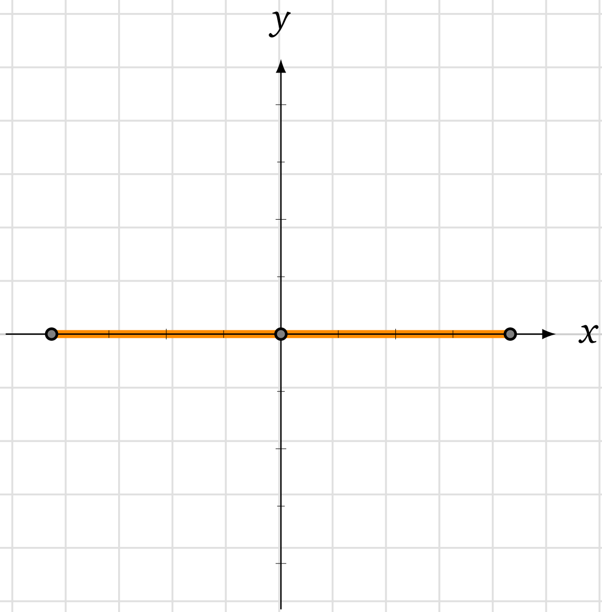


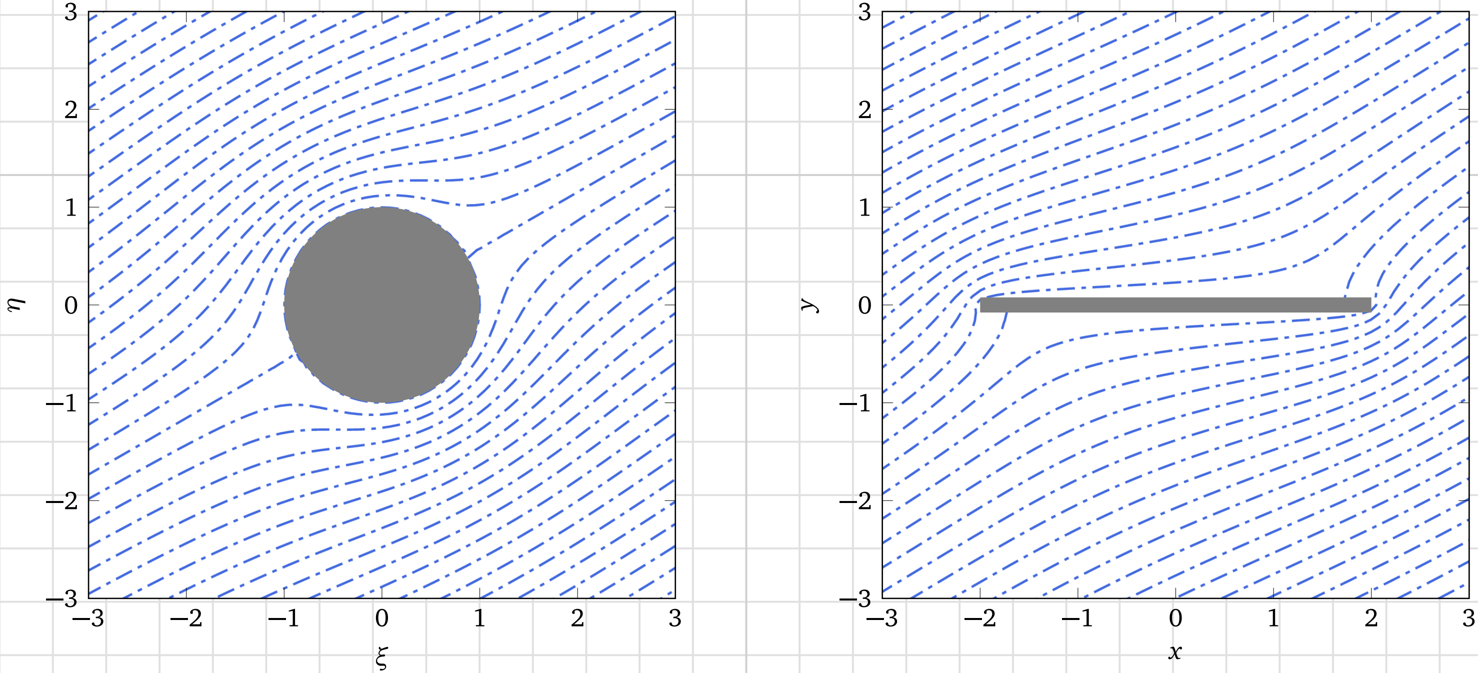


Joukowski transformation of the flow field

$$z = \zeta + \frac{a^2}{\zeta}$$

$$w(\zeta) = U_\infty \zeta e^{-i\alpha} + U_\infty \frac{a^2}{\zeta} e^{i\alpha}$$

$$w(z) = w(\zeta(z))$$

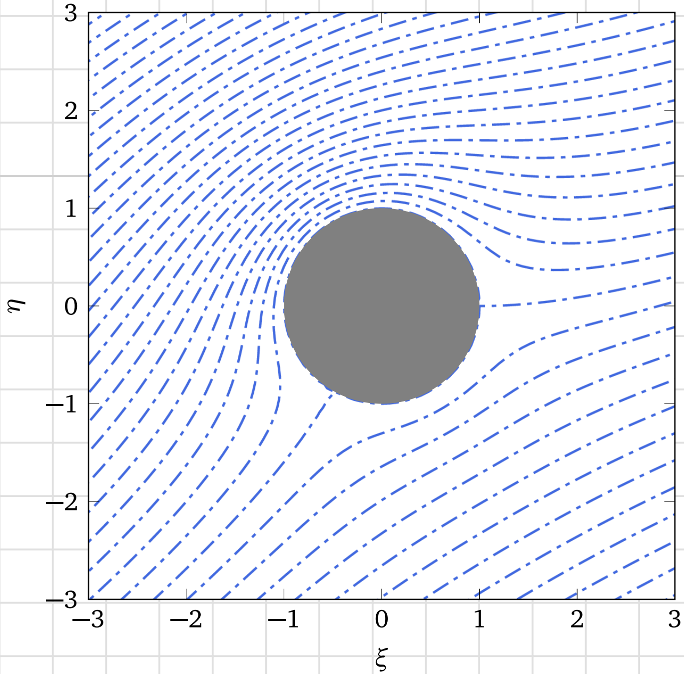


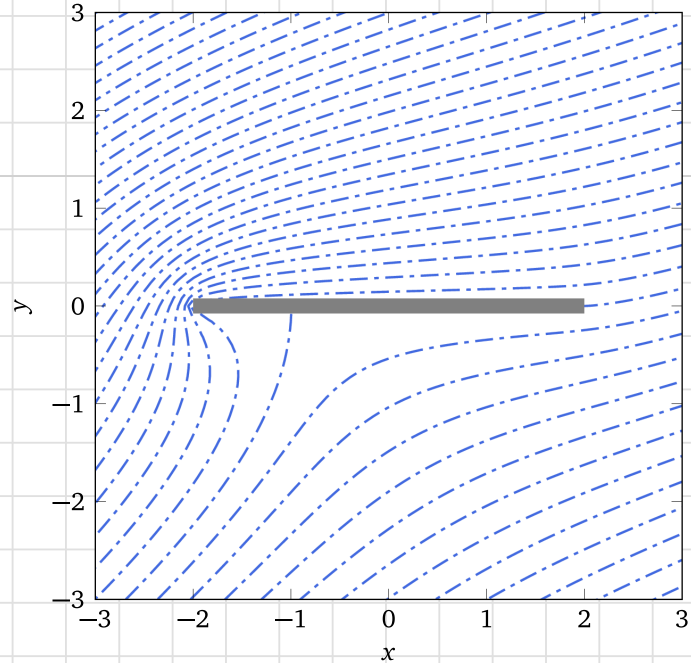
Kutta condition

$\Gamma = ?$

$$w(\zeta) = U_\infty \zeta e^{-i\alpha} + U_\infty \frac{a^2}{\zeta} e^{i\alpha} - \frac{i\Gamma}{2\pi} \ln \zeta$$

$$u_z - i v_z = \frac{dw}{dz} = \frac{dw}{d\zeta}\frac{d\zeta}{dz}$$

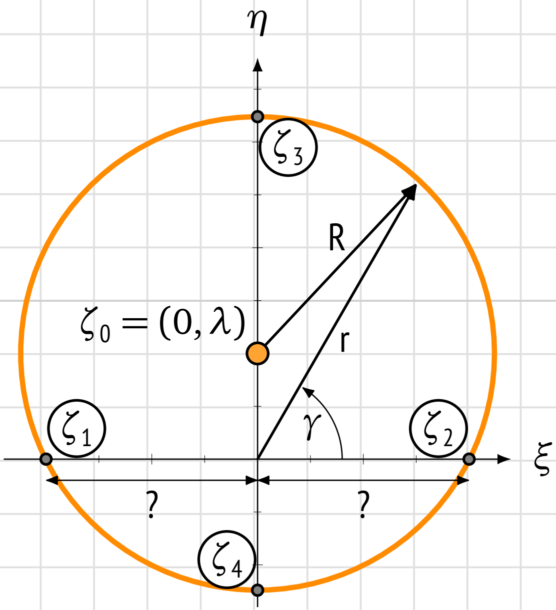


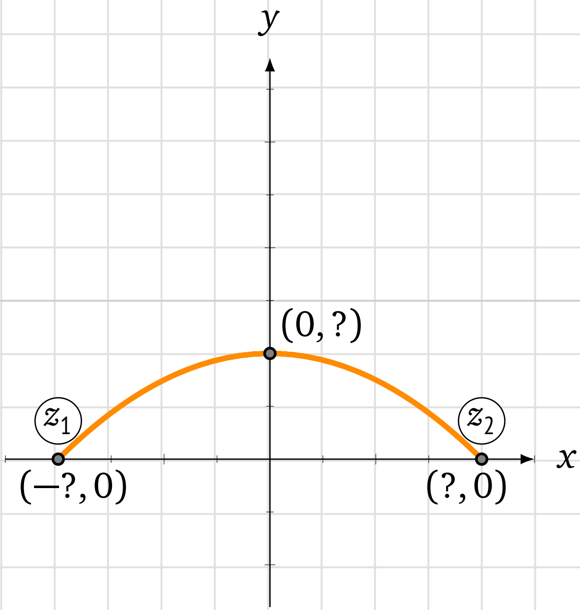


Kutta condition

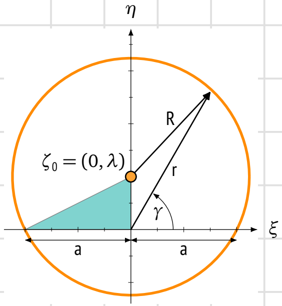
Joukowski airfoils with camber

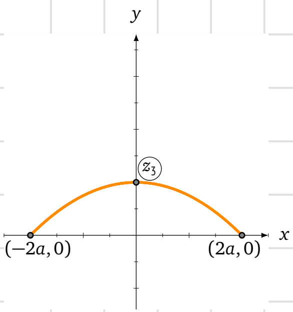
Now consider a circle with radius R that is shifted along the imaginary axis such that the centre of the circle is now at $\zeta_0 = (0, \lambda)$. The circle coordinates are $\zeta_{\text{circ}} = \lambda i + R e^{i\theta} = r(\gamma) e^{i\gamma}$





Joukowski airfoils with camber





Joukowski airfoils with camber

The complex potential of the flow around a cylinder with origin in ζ_0 and radius R in a uniform inflow with an angle of attack α can be written as:

$$w(\zeta) = U_\infty(\zeta - \zeta_0)e^{-i\alpha} + U_\infty \frac{R^2}{(\zeta - \zeta_0)} e^{i\alpha} - i\frac{\Gamma}{2\pi} \ln(\zeta - \zeta_0)$$

Joukowski airfoils with camber