EPFL

ME-445 AERODYNAMICS
Exercise 02
Week 1

Formula sheet

Cylindrical coordinates

$$\nabla\vec{u} = \left(\frac{\partial v_{\mathrm{r}}}{\partial r}, \frac{1}{r}\frac{\partial v_\theta}{\partial \theta}, 0\right)$$

$$\nabla\cdot\vec{u} = \frac{1}{r}\frac{\partial (r v_{\mathrm{r}})}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta}$$

$$\nabla\times\vec{u} = \left(0, 0, \frac{1}{r}\left[\frac{\partial (r v_\theta)}{\partial r} - \frac{\partial v_{\mathrm{r}}}{\partial \theta}\right]\right)$$

Potential flow

$$v_{\mathrm{r}} = \frac{\partial \phi}{\partial r} = \frac{1}{r}\frac{\partial \psi}{\partial \theta}, \quad v_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = -\frac{\partial \psi}{\partial r}$$

Uniform parallel flow $\quad w = \phi + i\psi = \mathrm{U}_\infty e^{-i\alpha} z$

Potential vortex in z_0 $\quad w = -\frac{i\gamma}{2\pi}\ln(z - z_0)$

Point source or sink in z_0 $\quad w = \frac{Q}{2\pi}\ln(z - z_0)$

Source-sink doublet in z_0 $\quad w = \frac{\mu}{2\pi(z - z_0)}$

$$\frac{\mathrm{d}w}{\mathrm{d}z} = u - iv$$

Milne-Thomson circle theorem:

$$g(z) = w(z) + \overline{w\left(\frac{a^2}{\bar{z}}\right)}$$

Thin airfoil theory

For a camber line with:

$$\frac{\mathrm{d}y_{\mathrm{c}}}{\mathrm{d}x} = A_0 + \sum_{n=1}^{\infty} A_{\mathrm{n}} \cos n\theta$$

$$\frac{x}{c} = \frac{(1 - \cos\theta)}{2}$$

we know:

$$k = 2\mathrm{U}_\infty\left[(\alpha - A_0)\frac{\cos\theta + 1}{\sin\theta} + \sum_{n=1}^{\infty} A_{\mathrm{n}}\sin n\theta\right]$$

$$A_0 = \frac{1}{\pi}\int_0^\pi \frac{\mathrm{d}y_{\mathrm{c}}}{\mathrm{d}x}\mathrm{d}\theta$$

$$A_{\mathrm{n}} = \frac{2}{\pi}\int_0^\pi \frac{\mathrm{d}y_{\mathrm{c}}}{\mathrm{d}x}\cos n\theta \mathrm{d}\theta$$

$$C_{\mathrm{l}} = 2\pi\alpha + \pi(A_1 - 2A_0)$$

$$C_{\mathrm{m,1/4}} = -\frac{\pi}{4}(A_1 - A_2)$$

$$x_{\mathrm{cp}} = \frac{1}{4} + \frac{\pi}{4C_{\mathrm{l}}}(A_1 - A_2)$$

Finite wings with AR=b^2/S

Sign convention:

if induced velocity points downward: $w(y) > 0$, $\alpha_{\mathrm{i}}(y) > 0$

if induced velocity points upward: $w < 0$, $\alpha_{\mathrm{i}} < 0$

Prandtl's lifting-line theory

$$\mathrm{U}_\infty\alpha_{\mathrm{i}}(y_0) = w(y_0) = -\frac{1}{4\pi}\int_{-b/2}^{b/2}\frac{(\mathrm{d}\Gamma/\mathrm{d}y)}{y - y_0}\mathrm{d}y$$

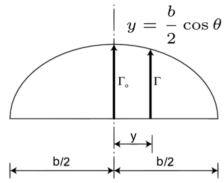
$$\alpha(y_0) = \alpha_{\mathrm{eff}}(y_0) + \alpha_{\mathrm{i}}(y_0)$$

Elliptical loading $\Gamma(y) = \Gamma_0\sqrt{1 - \left(\frac{2y}{b}\right)^2}$

$$w = \frac{\Gamma_0}{2b}$$

$$\alpha_{\mathrm{i}} = \frac{C_{\mathrm{L}}}{\pi \mathrm{AR}}$$

$$C_{\mathrm{D,i}} = \frac{C_{\mathrm{L}}^2}{\pi \mathrm{AR}}$$



General loading $\Gamma(\theta) = 2b\mathrm{U}_\infty\sum_{n=1}^{\infty} A_{\mathrm{n}}\sin n\theta$

$$w(\theta) = \mathrm{U}_\infty\sum_{n=1}^{\infty} nA_{\mathrm{n}}\frac{\sin n\theta}{\sin\theta}$$

$$C_{\mathrm{L}} = \pi A_1 \mathrm{AR}$$

$$C_{\mathrm{D,i}} = \frac{C_{\mathrm{L}}^2}{\pi \mathrm{AR}}(1 + \delta) \text{ with } \delta = \sum_{n=2}^{\infty} n\left(A_{\mathrm{n}}/A_1\right)^2$$

Boundary Layer

Flat plate **laminar** boundary layer

$\frac{\delta}{x} = \frac{5}{\sqrt{Re_{\mathrm{x}}}}$ boundary layer growth

$C_{\mathrm{f}} = \frac{1.328}{\sqrt{Re_{\mathrm{x}}}}$ skin friction drag coefficient

Flat plate **turbulent** boundary layer

$\frac{\delta}{x} = \frac{0.37}{Re_{\mathrm{x}}^{1/5}}$ boundary layer growth

$C_{\mathrm{f}} = \frac{0.074}{Re_{\mathrm{x}}^{1/5}}$ skin friction drag coefficient

Miscellanous

θ	$0°$	$30°$	$45°$	$60°$	$90°$
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0

water

kinematic viscosity	$\nu = 1 \times 10^{-6}\,\mathrm{m^2\,s^{-1}}$
density	$\rho = 1000\,\mathrm{kg\,m^{-3}}$

air

kinematic viscosity	$\nu = 1.5 \times 10^{-5}\,\mathrm{m^2\,s^{-1}}$
density	$\rho = 1.2\,\mathrm{kg\,m^{-3}}$

$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$$
$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$$

$$\cos 2\theta = 2\cos^2\theta - 1$$
$$\sin 2\theta = 2\sin\theta\cos\theta$$
$$\sin 3\theta = 3\sin\theta - 4\sin^3\theta$$
$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$$

$$\int_0^\pi \cos\theta \mathrm{d}\theta = 0$$
$$\int_0^\pi \sin\theta \mathrm{d}\theta = 2$$
$$\int_0^\pi \cos^2\theta \mathrm{d}\theta = \int_0^\pi \sin^2\theta \mathrm{d}\theta = \frac{\pi}{2}$$

$$\int_0^\pi \frac{\cos n\theta}{\cos\theta - \cos\theta_1}\mathrm{d}\theta = \pi\frac{\sin n\theta_1}{\sin\theta_1} \qquad n = 0, 1, 2, \ldots$$
$$\int_0^\pi \frac{\sin n\theta \sin\theta}{\cos\theta - \cos\theta_1}\mathrm{d}\theta = -\pi\cos n\theta_1 \qquad n = 1, 2, 3, \ldots$$

1. This tutorial is designed to take you through MATLAB tools that are useful to compute relevant aerodynamic variables and visualise flow. Consider a steady flow with a velocity field defined as:

$$\vec{V}(x,y) = \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} x \\ -y \end{pmatrix}$$

In this question, we wish to visualise the flow field.

It is convenient to express the flow field in the complex domain, in the form $\mathbf{V}(x,y) = u + iv$, to avoid creating several functions on MATLAB. You can create a function *velfield* that returns a complex array representing the flow field when given inputs x and y with the following code-line:

```
        velfield = @(x,y) x - y*1i;
```

(a) Use *meshgrid* to create arrays x and y that discretises a two dimensional space. It is advised you discretise space over the range 0 to 5 and that you use a step size of 0.1 in both directions. For help on using *meshgrid* you can type 'help meshgrid' in MATLAB's command window.

Solution:

```
dx = 0.1;
dy = 0.1;
[x,y] = meshgrid(0:dx:5,0:dy:5);
```

(b) Use *velfield* to compute a matrix W containing the velocity field in every point of your spatial field.

Solution:

```
W = velfield(x,y);
```

(c) You can retrieve two-dimensional velocity components U and V by splitting the real and imaginary part of the velocity field. Split your W array into real and imaginary part using functions *real* and *imag*.

Solution:

```
U=real(W);
V=imag(W);
```

(d) Use *quiver* to visualise the flow field. For help on using *quiver* you can type 'help quiver' in MATLAB's command window.

Solution:

```
figure
quiver(x,y,U,V)
axis equal tight
```

2. We are interested in finding out whether this flow is incompressible, in other words, if the divergence $\nabla \cdot \vec{V} = 0$ of the two-dimensional velocity field is zero.

   (a) Analytically calculate the divergence $\nabla \cdot \vec{V}$ of the investigated velocity field.

   Solution:

$$\begin{aligned} \nabla \cdot \vec{V} &= \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \\ &= 1 - 1 \\ &= 0 \end{aligned}$$

   (b) Use *gradient* to numerically compute the velocity field divergence. For help on using *gradient*, type 'help gradient' in the command window. It is advised you use the two-output syntax, as the gradients $\frac{\partial u}{\partial y}$ and $\frac{\partial v}{\partial x}$ will be of use later on.

   Solution:

```
[dudx,dudy]=gradient(U,dx);
[dvdx,dvdy]=gradient(V,dx);
divvelfield=dudx+dvdy;
```

   (c) Plot the velocity field divergence using *pcolor*. Does the figure look as expected based on your hand calculations? *Hint:* you can use the *shading flat* command to remove the gridlines. You can also add a colour bar with the *colorbar* command and set its range with the *caxis* command. Lastly, you can try decreasing the step size of your x and y grids from question 1(a).

   Solution:

```
figure,
pcolor(x,y,divvelfield)
shading flat
colorbar
caxis('auto')
```

   (d) Is this flow incompressible?

   Solution: Yes, the flow is incompressible as $\nabla \cdot \vec{V} = 0$.

3. In this question we introduce the stream function and visualise the flow streamlines. Since the flow is incompressible and $\nabla \cdot \vec{V} = 0$, we can introduce a scalar field called stream function ψ such that $\vec{V} = \nabla \times \vec{\psi}$, where $\vec{\psi} = (0, 0, \psi)$, as $\nabla \cdot (\nabla \times \vec{\psi})$ will always be zero.

   (a) Express the velocity components u and v in terms of ψ.

   Solution:

$$\vec{V} = \nabla \times \vec{\psi}$$

$$\begin{bmatrix} u \\ v \\ 0 \end{bmatrix} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & \psi \end{vmatrix}$$

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \frac{\partial \psi}{\partial y} \\ -\frac{\partial \psi}{\partial x} \end{bmatrix}$$

$$u = \frac{\partial \psi}{\partial y} \qquad v = -\frac{\partial \psi}{\partial x}$$

   (b) The stream function allows us to easily represent the flow topology. Integrate the partial derivatives you derived in the previous question to obtain the stream function using the boundary condition $\psi(0,0) = 0$.

   Solution:

$$\begin{cases} \psi &= \int u dy + C(x) \\ \psi &= -\int v dx + C(y) \end{cases}$$

$$\begin{cases} \psi &= xy + C(x) \\ \psi &= yx + C(y) \end{cases}$$

   so $C(x) = C(y) = C = 0$ as $\psi(0,0) = 0$. Therefore, $\psi = xy$.

   (c) Compute the stream function for the spatial domain.

   Solution:

```
psi = x.*y;
```

   (d) A streamline is defined such that:

$$\left.\frac{dy}{dx}\right|_{streamline} = \frac{v}{u}$$

   Show that $d\psi = 0$ along a streamline.

Solution:

$$\begin{aligned} d\psi &= \frac{\partial \psi}{\partial x}dx + \frac{\partial \psi}{\partial y}dy \\ &= -vdx + udy \end{aligned}$$

By definition, we know that along a streamline:

$$\begin{aligned} \frac{dy}{dx} &= \frac{v}{u} \\ udy - vdx &= 0 \end{aligned}$$

Therefore, along a streamline $d\psi = 0$.

(e) Use *contour* with $n = 30$ to display 30 stream function isolines. Based on your findings in the previous question, what do these isolines represent? For help on the *contour* function, you can type 'help contour' in the command window.

Solution:

```
figure
contour(x,y,psi,30), axis equal
```

The isolines represent streamlines, as ψ is constant along an isolines.

(f) Visualise the flow streamlines by using *streamslice*.

Solution:

```
figure
streamslice(x,y,U,V)
axis equal tight
```

4. In this question we are interested in finding out whether the velocity field is irrotational, or in other words if $\nabla \times \vec{V} = 0$.

   (a) Vorticity is a vector field defined as $\vec{\omega} = \nabla \times \vec{V}$. Derive the expression for the vorticity field.

   Solution:

$$\begin{aligned}\vec{\omega} &= \nabla \times \vec{V} \\ &= \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\vec{k} \\ &= (0-0)\vec{k} \\ &= 0\end{aligned}$$

   (b) Using your results from *gradient* in Question 2, numerically compute the vorticity field.

   Solution:

```
omega=dvdx-dudy;
```

   (c) Plot the vorticity field using *pcolor*. Does the figure look as expected, based on your hand calculations? You can use *shading flat* to remove the gridlines. You can also add a colour bar with *colorbar* and set its range with *caxis*. Lastly, you can try decreasing the step size of your x and y grids from question 1(a).

   Solution:

```
figure,
pcolor(x,y,omega)
shading flat
colorbar
caxis('auto')
axis xy equal tight
hold on
```

   (d) Is the flow irrotational?

   Solution: Yes, the flow is irrotational as $\nabla \times \vec{V} = 0$.

5. Since the flow is irrotational and $\boldsymbol{\nabla} \times \vec{V} = 0$, we can introduce a scalar field called velocity potential ϕ such that $\vec{V} = \nabla\phi$ as $\boldsymbol{\nabla} \times \nabla\phi$ will always be zero.

   (a) Express the velocity components u and v in terms of ϕ.

   Solution:

$$\vec{V} = \nabla\phi$$

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \frac{\partial\phi}{\partial x} \\ \frac{\partial\phi}{\partial y} \end{bmatrix}$$

$$u = \frac{\partial\phi}{\partial x} \qquad v = \frac{\partial\phi}{\partial y}$$

   (b) Integrate the partial differentials you derived in the previous question to obtain the velocity potential function using the boundary condition $\phi(0,0) = 0$.

   Solution:

$$\begin{cases} \phi &= \int u dx + C(y) \\ \phi &= \int v dy + C(x) \end{cases}$$

$$\begin{cases} \phi &= \frac{x^2}{2} + C(y) \\ \phi &= -\frac{y^2}{2} + C(x) \end{cases}$$

   Using the boundary condition, $\phi = \frac{x^2 - y^2}{2}$.

   (c) Compute the velocity potential function on your numerical spatial domain.

   Solution:

```
phi = (x.^2 - y.^2)/2;
```

   (d) Use *contour* with $n = 30$ to display 30 velocity potential isolines. For help on the *contour* function, you can type 'help contour' in the command window.

   Solution:

```
figure
contour(x,y,phi,30), axis equal
```

   (e) Compare your contour plot of the velocity potential with the contour plot of the stream function from Question 3. What do you observe?

   Solution: Equipotential lines are always perpendicular to streamlines.

6. We now consider another steady flow with a velocity field defined as:

$$\vec{V}(x,y) = \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -yx^2 \\ xy^2 \end{pmatrix}$$

(a) Adapt your script to visualise the velocity field

Solution:

```
velfield = @(x,y) - y.*x.^2 + 1i.*x.*y.^2;
dx = 0.1;
dy = 0.1;
[x,y] = meshgrid(0:dx:5,0:dy:5);
W = velfield(x,y);
U=real(W);
V=imag(W);

figure
quiver(x,y,U,V)
axis equal tight
```

(b) Use your script to show that this flow is incompressible.

Solution:

```
% Calculate and plot divergence of velocity field
[dudx,dudy]=gradient(U,dy);
[dvdx,dvdy]=gradient(V,dx);

divvelfield=dudx+dvdy;
figure,
pcolor(x,y,divvelfield)
shading flat
colorbar
caxis('auto')
```

(c) Calculate the stream function of this field using the boundary condition $\psi(0,0) = 0$.

Solution:

$$\begin{cases} \psi & = \int u dy + C(x) \\ \psi & = -\int v dx + C(y) \end{cases}$$

$$\begin{cases} \psi & = -0.5(xy)^2 + C(x) \\ \psi & = -0.5(yx)^2 + C(y) \end{cases}$$

so $C(x) = C(y) = C = 0$ as $\psi(0,0) = 0$. Therefore, $\psi = -0.5(yx)^2$.

(d) Adapt your script to visualise streamlines.

Solution:

```
psi = -0.5*(x.*y).^2;
figure
contour(x,y,psi,100), axis equal

% Compare to MATLAB streamlines using streamslice
figure
streamslice(x,y,U,V)
axis equal tight
```

(e) Use your script to visualise the vorticity field. Is this flow irrotational?

Solution:

```
omega=dvdx-dudy;
figure,
pcolor(x,y,omega), shading flat
colorbar
caxis('auto')
axis xy equal tight
hold on
```

No, the flow is not irrotational as $\boldsymbol{\nabla} \times \vec{V} \neq 0$.

7. In this question we are interested in calculating circulation around a rectangular path. The rectangle's vertices have coordinates (1,2), (3,2), (3,3) and (1,3). Circulation is defined as the surface integral of the vorticity field:

$$\Gamma = \iint_S \vec{\omega} \cdot \vec{n} \, dS$$

where $\vec{n}$ is the unit surface normal vector, such that for a two-dimensional surface:

$$\Gamma = \iint_S \omega_z \, dS$$

(a) Create an array with the x-coordinates of the vertices and an array with the y-coordinates of the vertices.

Solution:

```
x1 = 1;
y1 = 2;
x2 = 3;
y2 = 3;
xr = [x1 x2 x2 x1 x1];
yr = [y1 y1 y2 y2 y1];
```

(b) Visualise the rectangle using the *plot* function. *Hint:* you can use the *hold on* or *hold all* commands after plotting the vorticity field in the previous question to see the rectangle appear on top.

Solution:

```
plot(xr,yr)
```

(c) Analytically calculate circulation using a surface integral on the rectangular area.

Solution:

$$\begin{aligned} \vec{\omega} &= \boldsymbol{\nabla} \times \vec{V} \\ \omega_z &= \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \\ &= y^2 + x^2 \end{aligned}$$

$$
\begin{aligned}
\Gamma &= \iint_S \omega_z \, dS \\
&= \int_{y_1}^{y_2} \int_{x_1}^{x_2} x^2 + y^2 \, dx \, dy \\
&= \int_{y_1}^{y_2} \left[\frac{1}{3} x^3 + y^2 x \right]_1^3 dy \\
&= \int_2^3 \frac{26}{3} + 2y^2 dy \\
&= \left[\frac{26}{3} y + \frac{2}{3} y^3 \right]_2^3 \\
&= \frac{26}{3} + \frac{38}{3} = \frac{64}{3} = 21.33
\end{aligned}
$$

(d) We will now compute the same surface integral as in the previous question numerically. Use *inpolygon* to obtain the indices of all points of the x, y matrices inside the rectangle. For help on the *inpolygon* function, you can type 'help inpolygon' in the command window.

Solution:

```
in = inpolygon(x,y,xr,yr);
```

(e) Compute the surface integral to obtain the circulation *CircA* in the rectangle. *Hint:* you can approximate the integral $\iint_S \omega_z \, dS$ by a sum $\sum_{x_i} \sum_{y_i} \omega(x_i, y_i) \, \Delta x \Delta y$. Do you get the same result as in part (c)? If not, try increasing your spatial resolution.

Solution:

```
CircA = nansum(omega(in))*dx*dy;
```

With a step size of 0.1, CircA = 24.75. With a step size of 0.001, CircA = 21.37.

(f) According to Stokes theorem:

$$\iint_S \vec{\omega} \cdot \vec{n}\, dS = \iint_S \left(\boldsymbol{\nabla} \times \vec{V}\right) \cdot \vec{n}\, dS = \oint_c \vec{V} \cdot d\vec{l}$$

It can be useful to compute circulation using a path integral rather than a surface integral. It may be more accurate in case of poor spatial resolution and is computationally lighter. In this question we seek to calculate the circulation over the same rectangular path using a path integral.

Analytically compute circulation using a path integral on the same rectangular path. The rectangle's vertices have coordinates (1,2), (3,2), (3,3) and (1,3). Note that the line integral convention is anti-clockwise.

Solution:

$$\begin{aligned}
\Gamma &= \oint_c \vec{V} \cdot d\vec{l} \\
&= \int_{x_1}^{x_2} u(x, y = 2)dx + \int_{y_1}^{y_2} v(x = 3, y)dy + \int_{x_2}^{x_1} u(x, y = 3)dx + \int_{y_2}^{y_1} v(x = 1, y)dy \\
&= -\left[\frac{2}{3}x^3\right]_1^3 + \left[y^3\right]_2^3 - \left[x^3\right]_3^1 + \left[\frac{1}{3}y^3\right]_3^2 \\
&= -\frac{52}{3} + 26 + 19 - \frac{19}{3} \\
&= 21.33
\end{aligned}$$

(g) Numerically compute the circulation *CircR* along the rectangular path using *trapz*.

Solution:

```
idx1 = x1/dx+1;
idx2 = x2/dx+1;
idy1 = y1/dy+1;
idy2 = y2/dy+1;

% Integrate v.dl over contour
CircR = trapz(x1:dx:x2,U(idy1,idx1:idx2)) + ...
trapz(y1:dy:y2,V(idy1:idy2,idx2)) - ...
trapz(x1:dx:x2,U(idy2,idx1:idx2)) - ...
trapz(y1:dy:y2,V(idy1:idy2,idx1));
```

With a step size of 0.1, CircR = 21.34. With a step size of 0.001, CircR = 21.33.

(h) Are the values for CircR and CircA the same? If not, why? What if you increase your spatial resolution by setting the meshgrid step to 0.001?

Solution: Error is a result of limited spatial resolution and numerical rounding errors. For a step size of 0.001, the error is smaller than one percent when calculating circulation with either the surface integral of the path integral.