ME-445 AERODYNAMICS
Exercise 03
Week 2

Formula sheet

Cylindrical coordinates

$$\nabla \vec{u} = \left(\frac{\partial v_{\mathrm{r}}}{\partial r}, \frac{1}{r}\frac{\partial v_\theta}{\partial \theta}, 0 \right)$$

$$\nabla \cdot \vec{u} = \frac{1}{r}\frac{\partial (r v_{\mathrm{r}})}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta}$$

$$\nabla \times \vec{u} = \left(0, 0, \frac{1}{r}\left[\frac{\partial (r v_\theta)}{\partial r} - \frac{\partial v_{\mathrm{r}}}{\partial \theta} \right] \right)$$

Potential flow

$$v_{\mathrm{r}} = \frac{\partial \phi}{\partial r} = \frac{1}{r}\frac{\partial \psi}{\partial \theta}, \quad v_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = -\frac{\partial \psi}{\partial r}$$

Uniform parallel flow $\quad w = \phi + i\psi = \mathrm{U}_\infty e^{-i\alpha}\, z$

Potential vortex in z_0 $\quad w = -\dfrac{i\gamma}{2\pi}\ln(z - z_0)$

Point source or sink in z_0 $\quad w = \dfrac{Q}{2\pi}\ln(z - z_0)$

Source-sink doublet in z_0 $\quad w = \dfrac{\mu}{2\pi(z - z_0)}$

$$\frac{\mathrm{d}w}{\mathrm{d}z} = u - iv$$

Milne-Thomson circle theorem:

$$g(z) = w(z) + \overline{w\left(\frac{a^2}{\bar{z}}\right)}$$

Thin airfoil theory

For a camber line with:

$$\frac{\mathrm{d}y_{\mathrm{c}}}{\mathrm{d}x} = A_0 + \sum_{n=1}^{\infty} A_{\mathrm{n}} \cos n\theta$$

$$\frac{x}{c} = \frac{(1 - \cos\theta)}{2}$$

we know:

$$k = 2\mathrm{U}_\infty \left[(\alpha - A_0)\frac{\cos\theta + 1}{\sin\theta} + \sum_{n=1}^{\infty} A_{\mathrm{n}} \sin n\theta \right]$$

$$A_0 = \frac{1}{\pi}\int_0^\pi \frac{\mathrm{d}y_{\mathrm{c}}}{\mathrm{d}x}\mathrm{d}\theta$$

$$A_{\mathrm{n}} = \frac{2}{\pi}\int_0^\pi \frac{\mathrm{d}y_{\mathrm{c}}}{\mathrm{d}x}\cos n\theta \mathrm{d}\theta$$

$$C_{\mathrm{l}} = 2\pi\alpha + \pi(A_1 - 2A_0)$$

$$C_{\mathrm{m,1/4}} = -\frac{\pi}{4}(A_1 - A_2)$$

$$x_{\mathrm{cp}} = \frac{1}{4} + \frac{\pi}{4C_{\mathrm{l}}}(A_1 - A_2)$$

Finite wings with AR=b^2/S

Sign convention:

if induced velocity points downward: $w(y) > 0$, $\alpha_{\mathrm{i}}(y) > 0$

if induced velocity points upward: $w < 0$, $\alpha_{\mathrm{i}} < 0$

Prandtl's lifting-line theory

$$\mathrm{U}_\infty \alpha_{\mathrm{i}}(y_0) = w(y_0) = -\frac{1}{4\pi}\int_{-b/2}^{b/2} \frac{(\mathrm{d}\Gamma/\mathrm{d}y)}{y - y_0}\mathrm{d}y$$

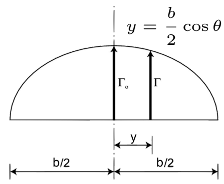
$$\alpha(y_0) = \alpha_{\mathrm{eff}}(y_0) + \alpha_{\mathrm{i}}(y_0)$$

Elliptical loading $\Gamma(y) = \Gamma_0\sqrt{1 - \left(\dfrac{2y}{b}\right)^2}$

$$w = \frac{\Gamma_0}{2b}$$

$$\alpha_{\mathrm{i}} = \frac{C_{\mathrm{L}}}{\pi \mathrm{AR}}$$

$$C_{\mathrm{D,i}} = \frac{C_{\mathrm{L}}^2}{\pi \mathrm{AR}}$$



General loading $\Gamma(\theta) = 2b\mathrm{U}_\infty \sum_{n=1}^{\infty} A_{\mathrm{n}} \sin n\theta$

$$w(\theta) = \mathrm{U}_\infty \sum_{n=1}^{\infty} n A_{\mathrm{n}} \frac{\sin n\theta}{\sin\theta}$$

$$C_{\mathrm{L}} = \pi A_1\, \mathrm{AR}$$

$$C_{\mathrm{D,i}} = \frac{C_{\mathrm{L}}^2}{\pi \mathrm{AR}}(1 + \delta) \text{ with } \delta = \sum_{n=2}^{\infty} n\,(A_{\mathrm{n}}/A_1)^2$$

Boundary Layer

Flat plate **laminar** boundary layer

$\dfrac{\delta}{x} = \dfrac{5}{\sqrt{Re_{\mathrm{x}}}}$ boundary layer growth

$C_{\mathrm{f}} = \dfrac{1.328}{\sqrt{Re_{\mathrm{x}}}}$ skin friction drag coefficient

Flat plate **turbulent** boundary layer

$\dfrac{\delta}{x} = \dfrac{0.37}{Re_{\mathrm{x}}^{1/5}}$ boundary layer growth

$C_{\mathrm{f}} = \dfrac{0.074}{Re_{\mathrm{x}}^{1/5}}$ skin friction drag coefficient

Miscellanous

θ	$0°$	$30°$	$45°$	$60°$	$90°$
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0

water

kinematic viscosity $\nu = 1 \times 10^{-6}\,\mathrm{m^2\,s^{-1}}$

density $\rho = 1000\,\mathrm{kg\,m^{-3}}$

air

kinematic viscosity $\nu = 1.5 \times 10^{-5}\,\mathrm{m^2\,s^{-1}}$

density $\rho = 1.2\,\mathrm{kg\,m^{-3}}$

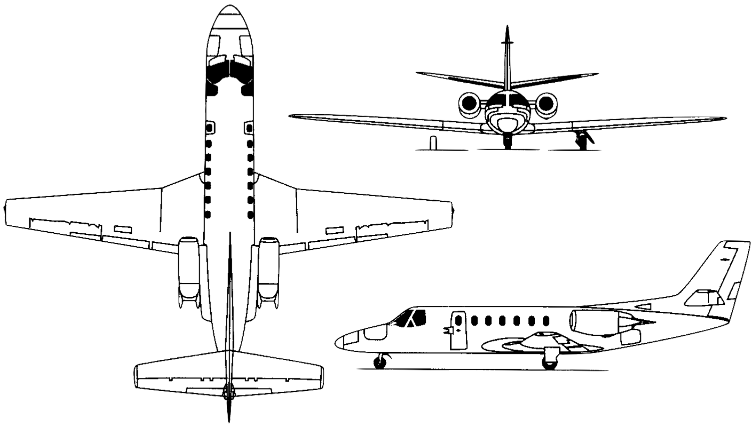
$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$$
$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$$

$$\cos 2\theta = 2\cos^2\theta - 1$$
$$\sin 2\theta = 2\sin\theta\cos\theta$$
$$\sin 3\theta = 3\sin\theta - 4\sin^3\theta$$
$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$$

$$\int_0^\pi \cos\theta \mathrm{d}\theta = 0$$
$$\int_0^\pi \sin\theta \mathrm{d}\theta = 2$$
$$\int_0^\pi \cos^2\theta \mathrm{d}\theta = \int_0^\pi \sin^2\theta \mathrm{d}\theta = \frac{\pi}{2}$$

$$\int_0^\pi \frac{\cos n\theta}{\cos\theta - \cos\theta_1}\mathrm{d}\theta = \pi\frac{\sin n\theta_1}{\sin\theta_1} \qquad n = 0, 1, 2, \ldots$$
$$\int_0^\pi \frac{\sin n\theta \sin\theta}{\cos\theta - \cos\theta_1}\mathrm{d}\theta = -\pi\cos n\theta_1 \qquad n = 1, 2, 3, \ldots$$

1. Consider an executive jet transport patterned after the Cessna 560 Citation V. The airplane is cruising at a velocity of $790\,\mathrm{km\,h^{-1}}$ at an altitude of $10\,\mathrm{km}$, where the ambient air density is $0.41\,\mathrm{kg\,m^{-3}}$. The weight and wing planform areas of the airplane are $6800\,\mathrm{kg}$ and $32\,\mathrm{m^2}$, respectively. The drag coefficient at cruise is 0.015.



   (a) Calculate the lift coefficient and the lift-to-drag ratio at cruise flight.

   Solution:

$$u_\infty = 790\,\mathrm{km\,h^{-1}} = 790 \cdot \frac{1000\,\mathrm{m}}{3600\,\mathrm{s}} = 219.4\,\mathrm{m\,s^{-1}}$$

$$C_L = \frac{L}{\frac{1}{2}\rho u_\infty^2 S} = \frac{Mg}{\frac{1}{2}\rho u_\infty^2 S} = \frac{6800\,\mathrm{kg}\cdot 10\,\mathrm{m\,s^{-2}}}{\frac{1}{2}\cdot 0.41\,\mathrm{kg\,m^{-3}}\cdot(219.4\,\mathrm{m\,s^{-1}})^2\cdot 32\,\mathrm{m^2}} = 0.215$$

$$\frac{C_L}{C_D} = \frac{0.215}{0.015} = 14.35$$

   (b) Assume that the airplane takes-off when the speed is $240\,\mathrm{km\,h^{-1}}$. It has a minimum steady flight speed of $160\,\mathrm{km\,h^{-1}}$ at standard conditions, below this velocity the airplane will stall and can no longer maintain its altitude. The maximum take-off weight is $7200\,\mathrm{kg}$. The ambient air density at standard sea level is $1.22\,\mathrm{kg\,m^{-3}}$. Find the value of the maximum lift coefficient for the airplane.

   Solution:

$$u_{\text{take-off}} = 240\,\mathrm{km\,h^{-1}} = 240 \cdot \frac{1000\,\mathrm{m}}{3600\,\mathrm{s}} = 66.7\,\mathrm{m\,s^{-1}}$$

$$u_{\text{stall}} = 160\,\mathrm{km\,h^{-1}} = 160 \cdot \frac{1000\,\mathrm{m}}{3600\,\mathrm{s}} = 44.4\,\mathrm{m\,s^{-1}}$$

$$C_{L\,\text{max}} = \frac{L}{\frac{1}{2}\rho u_\infty^2 S} = \frac{Mg}{\frac{1}{2}\rho u_\infty^2 S} = \frac{7200\,\mathrm{kg}\cdot 10\,\mathrm{m\,s^{-2}}}{\frac{1}{2}\cdot 1.22\,\mathrm{kg\,m^{-3}}\cdot(44.4\,\mathrm{m\,s^{-1}})^2\cdot 32\,\mathrm{m^2}} = 1.867$$

(c) Geneva airport's concrete runway is the longest in Switzerland, with a length of $3900\,\mathrm{m}$. The Cessna 560 Citation V has two JT15D5D engines producing $13.6\,\mathrm{kN}$ of net thrust force T_n each, resulting in constant acceleration of the aircraft along the runway. Assuming the same take-off speed and air density as in the previous question, calculate the proportion of the runway the executive jet will require to take off.

Solution:

Assuming constant acceleration of the airplane $a = \frac{\sum F}{M}$, we can compute time taken to reach take off speed $t_{\text{take-off}}$ as follows:

$$
\begin{aligned}
t_{\text{take-off}} \cdot a &= u_{\text{take-off}} \\
&= \frac{u_{\text{take-off}}}{a} \\
&= \frac{M \cdot u_{\text{take-off}}}{2 \cdot T_n} \\
&= \frac{7200\,\mathrm{kg} \cdot 66.7\,\mathrm{m\,s^{-1}}}{2 \cdot 13.6\,\mathrm{kN}} \\
&= 17.7\,\mathrm{s}
\end{aligned}
$$

The average velocity of the airplane on the runway is given by:

$$
\begin{aligned}
\bar{u}_{\text{runway}} &= 0.5 \cdot a \cdot t_{\text{take-off}} \\
&= 0.5 \cdot u_{\text{take-off}} \\
&= 33.3\,\mathrm{m\,s^{-1}}
\end{aligned}
$$

Thus, the take-off distance is given by:

$$
d_{\text{take-off}} = \bar{u}_{\text{runway}} \cdot t_{\text{take-off}} = 590\,\mathrm{m}
$$

The proportion of the runway used by this airplane to take-off is $\dfrac{590\,\mathrm{m}}{3900\,\mathrm{m}} \cdot 100 = 15\%$.

(d) On a warm summer day, Geneva's air density is $1.10\,\mathrm{kg\,m^{-3}}$. Calculate the proportion of the runway the executive jet will require to take off on this warm summer day.

Solution:

First we compute the lift coefficient at take-off in standard conditions:

$$C_{L\,\text{take-off}} = \frac{Mg}{\frac{1}{2}\rho u_\infty^2 S} = \frac{7200\,\mathrm{kg}\cdot 10\,\mathrm{m\,s^{-2}}}{\frac{1}{2}\cdot 1.22\,\mathrm{kg\,m^{-3}}\cdot(66.7\,\mathrm{m\,s^{-1}})^2\cdot 32\,\mathrm{m^2}} = 0.83$$

We can calculate the new take-off velocity using the lift coefficient and air density:

$$u_\infty = \sqrt{\frac{Mg}{\frac{1}{2}\rho C_{L\,\text{take-off}} S}} = \sqrt{\frac{7200\,\mathrm{kg}\cdot 10\,\mathrm{m\,s^{-2}}}{\frac{1}{2}\cdot 1.10\,\mathrm{kg\,m^{-3}}\cdot 0.83\cdot 32\,\mathrm{m^2}}} = 70.2\,\mathrm{m\,s^{-1}}$$

Assuming constant acceleration of the airplane $a = \frac{\sum F}{M}$, we can compute time taken to reach take off speed $t_{\text{take-off}}$ as follows:

$$\begin{aligned}
t_{\text{take-off}}\cdot a &= u_{\text{take-off}}\\
&= \frac{u_{\text{take-off}}}{a}\\
&= \frac{M\cdot u_{\text{take-off}}}{2\cdot T_n}\\
&= \frac{7200\,\mathrm{kg}\cdot 70.2\,\mathrm{m\,s^{-1}}}{2\cdot 13.6\,\mathrm{kN}}\\
&= 18.6\,\mathrm{s}
\end{aligned}$$

The average velocity of the airplane on the runway is given by:

$$\begin{aligned}
\bar{u}_{\text{runway}} &= 0.5\cdot a\cdot t_{\text{take-off}}\\
&= 0.5\cdot u_{\text{take-off}}\\
&= 35.1\,\mathrm{m\,s^{-1}}
\end{aligned}$$

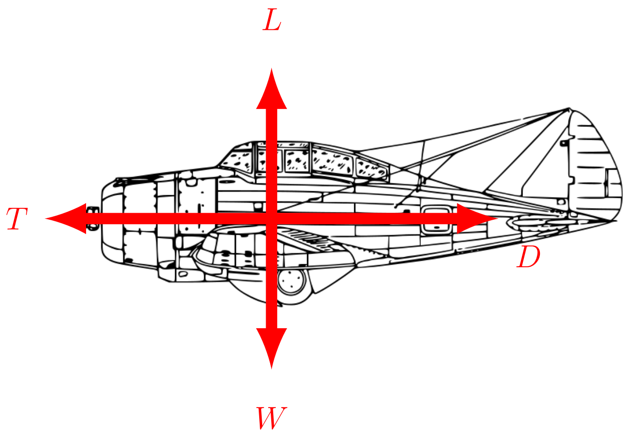
Thus, the take-off distance is given by:

$$d_{\text{take-off}} = \bar{u}_{\text{runway}}\cdot t_{\text{take-off}} = 653\,\mathrm{m}$$

The proportion of the runway used by this airplane to take-off is $\dfrac{653\,\mathrm{m}}{3900\,\mathrm{m}}\cdot 100 = 17\%$.

2. Consider a Seversky P-35. The wing planform area and the gross weight of the fighter aircraft are $20.5\,\mathrm{m}^2$ and $25\,\mathrm{kN}$, respectively.

   (a) Calculate the power required for the aircraft to fly in steady level flight with $C_L = 0.15$ and $C_D = 0.0275$ at standard conditions ($\rho = 1.225\,\mathrm{kg\,m^{-3}}$).



Solution:

$$
\begin{aligned}
P &= Tu_\infty = Du_\infty \\
W &= L = \frac{1}{2}\rho_\infty u_\infty^2 S C_L \\
u_\infty &= \sqrt{\frac{2W}{\rho_\infty S C_L}} = \sqrt{\frac{2 \cdot 25\,000\,\mathrm{N}}{1.225\,\mathrm{kg\,m^{-3}} \cdot 20.5\,\mathrm{m^2} \cdot 0.15}} = 115.2\,\mathrm{m\,s^{-1}} \\
P &= Dv_\infty = \frac{1}{2}\rho_\infty u_\infty^2 S C_D \cdot u_\infty = \frac{1}{2} \cdot 1.225\,\mathrm{kg\,m^{-3}} \cdot (115.2\,\mathrm{m\,s^{-1}})^3 \cdot 20.5\,\mathrm{m^2} \cdot 0.0275 = \\
&= 5.28 \times 10^5\mathrm{N\,m\,s^{-1}} = 528\,\mathrm{kW}
\end{aligned}
$$

   (b) A important performance characteristics of an airplane is its maximum rate-of-climb B. The rate-of-climb is the increase in altitude per unit of time and it is proportional to the difference in maximum power available from the engine and the power required by the airplane to overcome aerodynamic drag. This difference is referred to as the excess power:

$$B = \frac{\text{excess power}}{W} \quad ,$$

where W is the weight of the airplane.
Calculate B for the P-35 fighter equipped with a Pratt & Whitney R-1830-45 engine rated at $788\,\mathrm{kW}$.

Solution:

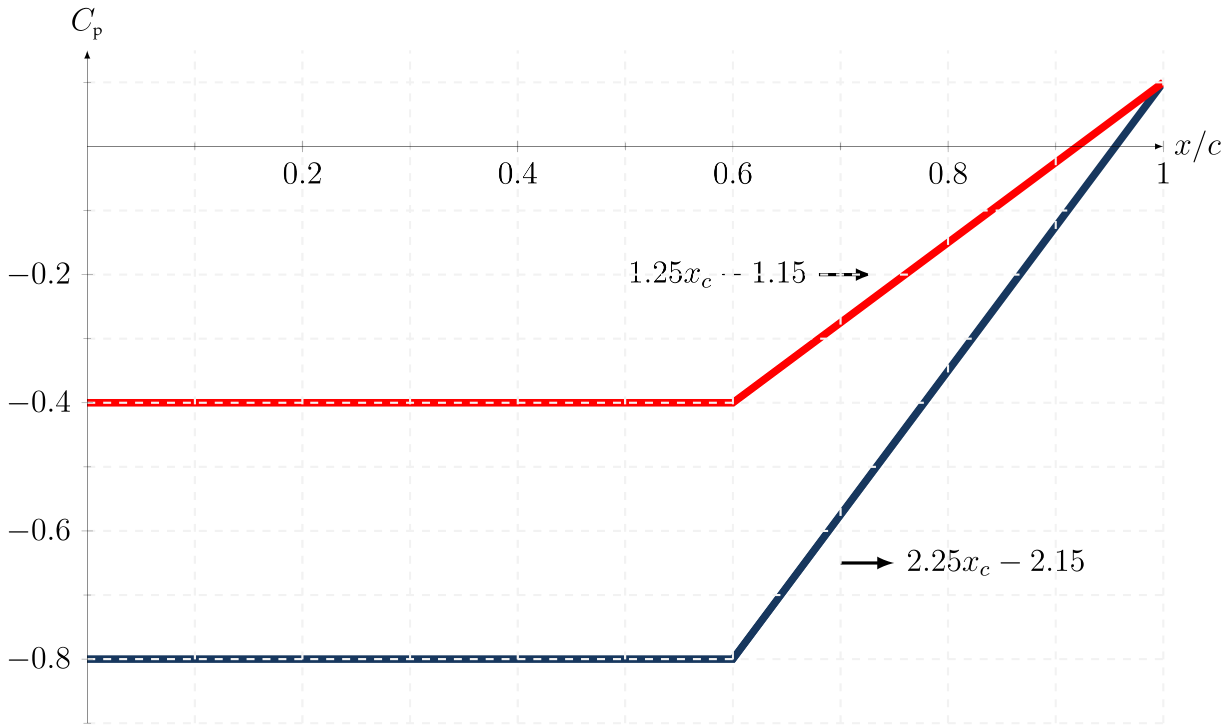
$$
\text{excess power} = P_{\text{engine}} - P = 788 \times 10^3\,\mathrm{W} - 528 \times 10^3\,\mathrm{W} = 260 \times 10^4\,\mathrm{W} = 260\,\mathrm{kW}
$$

$$
B = \frac{260 \times 10^3\,\mathrm{W}}{25 \times 10^3\,\mathrm{N}} = 10.4\,\mathrm{m\,s^{-1}} = 624\,\mathrm{m\,min^{-1}}
$$

3. The pressure distribution over a section of a 2D wing at $4°$ of incidence may be approximated as follows:

   - **suction side**: c_p constant at -0.8 from the leading edge to $60\,\%$ chord, then increasing linearly to 0.1 at the trailing edge
   - **pressure side**: c_p constant at -0.4 from the leading edge to $60\,\%$ chord, then increasing linearly to 0.1 at the trailing edge.

   (a) Draw the pressure distribution.



   (b) Estimate the lift coefficient and the pitching moment coefficient about the leading edge due to lift.

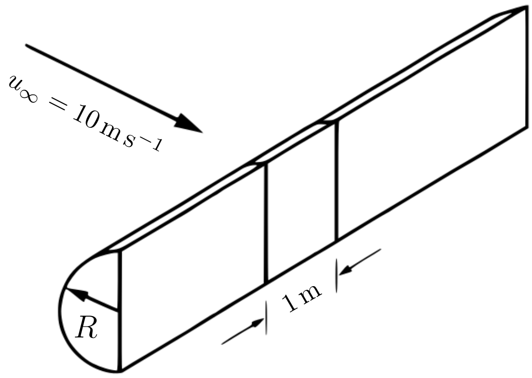
Solution: using $\dfrac{x}{c} = x_c$

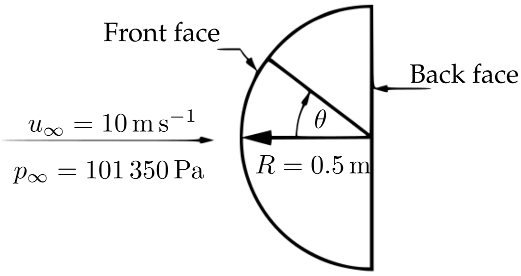
$$
\begin{aligned}
c_n &= \int\limits_0^1 (c_{p\,\mathrm{PS}} - c_{p\,\mathrm{SS}}) dx_c \\
&= \int\limits_0^{0.6} (-0.4 - (-0.8)) dx_c + \int\limits_{0.6}^{1} (1.25x_c - 1.15 - 2.25x_c - 2.15) dx_c \\
&= \int\limits_0^{0.6} 0.4 dx_c + \int\limits_{0.6}^{1} (-x_c + 1) dx_c \\
&= 0.4x_c \Big|_0^{0.6} - \frac{1}{2}x_c^2 \Big|_{0.6}^{1} + x_c \Big|_{0.6}^{1} \\
&= 0.24 - 0.32 + 0.4 = 0.32
\end{aligned}
$$

$$
c_l = c_n \cos\alpha = 0.319
$$

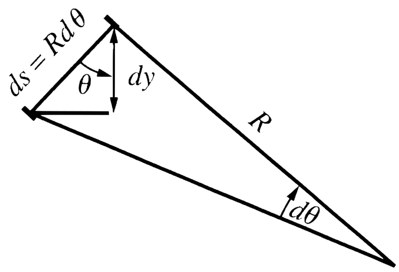
$$
\begin{aligned}
c_{m\,\mathrm{LE}} &= \int\limits_0^1 (c_{p\,\mathrm{SS}} - c_{p\,\mathrm{PS}}) x_c dx_c \\
&= -\int\limits_{0.6}^{1} (-x_c + 1) x_c dx_c - \int\limits_0^{0.6} 0.4 x_c dx_c \\
&= \frac{1}{3}x_c^3 \Big|_{0.6}^{1} - \frac{1}{2}x_c^2 \Big|_{0.6}^{1} - \frac{0.4}{2}x_c^2 \Big|_0^{0.6} \\
&= 0.261 - 0.320 - 0.072 = -0.131
\end{aligned}
$$

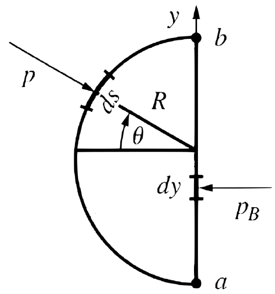
4. Consider a long dowel with a semicircular cross section. The dowel is immersed in a flow of air, with its axis perpendicular to the flow. The rounded section of the dowel is facing the flow. We call this rounded section the *front face* of the dowel. The radius of the semicircular cross section is $R = 0.5\,\mathrm{m}$. The velocity of the flow upstream of the dowel is $u_\infty = 10\,\mathrm{m\,s^{-1}}$. Assume inviscid flow. The pressure and the velocity of the flow along the surface of the rounded front face of the dowel are a function of the location along the surface, denoted by angle θ. Along the front rounded surface $v = v(\theta) = 2u_\infty \sin\theta$ and p varies accordingly. On the flat back face, the pressure is constant and equal to $p_B = p_\infty - 0.7\rho_\infty u_\infty^2$. The free-stream density is $\rho_\infty = 1.225\,\mathrm{kg\,m^{-3}}$. Calculate the aerodynamic force per unit depth exerted by the surface pressure distribution on 2D the dowel.





Solution: Due to the symmetry of the semicircular cross section there is no net force on the cross section in the direction perpendicular to the free stream. The force due to the pressure pushing down on the upper surface is exactly cancelled by the equal and opposite force due to the pressure pushing up on the lower surface.





From the geometry we have:

$$
\begin{aligned}
ds &= Rd\theta \\
dy &= ds\cos\theta
\end{aligned}
$$

The horizontal force is:

$$F_{||} = (pds)\cos\theta = pR\cos\theta d\theta \quad .$$

The total horizontal force d_f exerted by the pressure distribution on the rounded front face is:

$$d_f = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} pR\cos\theta d\theta \quad ,$$

where p is obtained from Bernoulli's equation $p = p_\infty + \frac{1}{2}\rho(u_\infty^2 - v^2)$.

$$\begin{aligned}
d_f &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[p_\infty + \frac{1}{2}\rho(u_\infty^2 - v^2)\right] R\cos\theta d\theta\bigg|_{v=2u_\infty \sin\theta} = \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[p_\infty + \frac{1}{2}\rho(u_\infty^2 - (2u_\infty\sin\theta)^2)\right] R\cos\theta d\theta = \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[p_\infty + \frac{1}{2}\rho u_\infty^2(1 - 4\sin\theta^2)\right] R\cos\theta d\theta.
\end{aligned}$$

The pressure along the back face is constant and equal to p_B and the horizontal force acting on the back face in the negative x-direction is

$$\begin{aligned}
d_b &= -2Rp_B \\
&= -\int_a^b p_B dy\bigg|_{dy=R\cos\theta d\theta} \\
&= -\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} p_B R\cos\theta d\theta \\
&= -\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[p_\infty - 0.7\rho u_\infty^2\right] R\cos\theta d\theta.
\end{aligned}$$

The resultant aerodynamic force on the cross section is:

$$\begin{aligned}
d &= d_f + d_b \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[p_\infty + \frac{1}{2}\rho u_\infty^2(1 - 4\sin\theta^2)\right] R\cos\theta d\theta - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[p_\infty - 0.7\rho u_\infty^2\right] R\cos\theta d\theta \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[\left(\frac{1}{2} + 0.7\right)\rho u_\infty^2 - 2\rho u_\infty^2 \sin\theta^2\right] R\cos\theta d\theta \\
&= 1.2\rho u_\infty^2 R \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\theta d\theta - 2\rho u_\infty^2 R \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin\theta^2 cos\theta d\theta \\
&= 2.4\rho u_\infty^2 R - 2\rho u_\infty^2 R\left(\frac{1}{3} + \frac{1}{3}\right) \\
&= 1.067\rho u_\infty^2 R \\
&= 1.067 \cdot 1.225\,\mathrm{kg\,m^{-3}} \cdot (10\mathrm{m\,s^{-1}})^2 \cdot 0.5\,\mathrm{m} \\
&= 65.35\,\mathrm{N\,m^{-1}}
\end{aligned}$$

* Note that the force is in $\mathrm{N\,m^{-1}}$ as we considered a 2D dowel.