Exercise sheet 4 - Annexe - Mechanical Vibrations

Solutions

1 Airfoil on two springs

1. The system has two degrees of freedom: a vertical translation and a rotation around the point O. The springs elongation are given by $\Delta L_{k_1} = x - l_1\theta$ and $\Delta L_{k_2} = x + l_2\theta$. Thus, the equations of motion read:

$$\begin{aligned} m\ddot{x} + (k_1 + k_2)x - (k_1 l_1 - k_2 l_2)\theta = 0 \\ mr^2\ddot{\theta} + (k_1 l_1^2 + k_2 l_2^2)\theta - (k_1 l_1 - k_2 l_2)x = 0 \end{aligned} \tag{1}$$

which, in matrix form, can be rewritten as:

$$\begin{bmatrix} m & 0 \\ 0 & mr^2 \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} k_1 + k_2 & -(k_1 l_1 - k_2 l_2) \\ -(k_1 l_1 - k_2 l_2) & (k_1 l_1^2 + k_2 l_2^2) \end{bmatrix} \begin{bmatrix} x \\ \theta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{2}$$

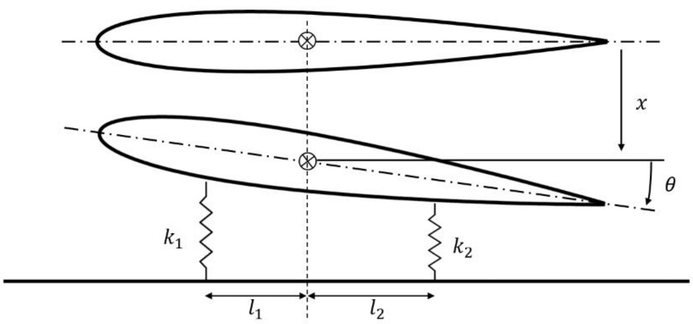

Figure 1: Oscillating airfoil.

2. From equation 2, one can easily see that the condition for the translation and rotation to be uncoupled is met if $(k_1 l_1 - k_2 l_2) = 0$, or:

$$\frac{k_1}{k_2} = \frac{l_2}{l_1} \tag{3}$$

3. Assuming a motion in the form $x(t) = Xe^{i\omega t}$ and $\theta(t) = \Theta e^{i\omega t}$, equation 2 can be rewritten as:

$$\begin{bmatrix} -\omega^2 m + k_1 + k_2 & -(k_1 l_1 - k_2 l_2) \\ -(k_1 l_1 - k_2 l_2) & -\omega^2 mr^2 + (k_1 l_1^2 + k_2 l_2^2) \end{bmatrix} \begin{bmatrix} X \\ \Theta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{4}$$

Non-trivial solutions are found if the determinant of equation 4 is equal to 0. The characteristic equation of the matrix is given by:

$$\left(-\omega^2 + \frac{k_1 + k_2}{m}\right)\left(-\omega^2 + \frac{k_1 l_1^2 + k_2 l_2^2}{mr^2}\right) - \frac{(k_1 l_1 - k_2 l_2)^2}{m^2 r^2} = 0$$
$$\Leftrightarrow (-\omega^2 + A)(-\omega^2 + B) - C = 0 \tag{5}$$

And so,

$$\omega_1^2 = \frac{A+B}{2} - \sqrt{\left(\frac{A-B}{2}\right)^2 + C}$$
$$\omega_2^2 = \frac{A+B}{2} + \sqrt{\left(\frac{A-B}{2}\right)^2 + C} \tag{6}$$

4. Using the state space representation, the equation of motion can readily be integrated in time and solved. One can for instance use *ode45* from Matlab or *scipy.integrate.solve_ivp* from Python to perform the time integration.

   For $l_2 = l_{2,decoupled}$ with $x_0 = 0.01$ m, $\theta_0 = 0.006$ rad, $\dot{x}_0 = 0$ m/s and $\dot{\theta}_0 = 0$ rad/s, we have the following decoupled motions:

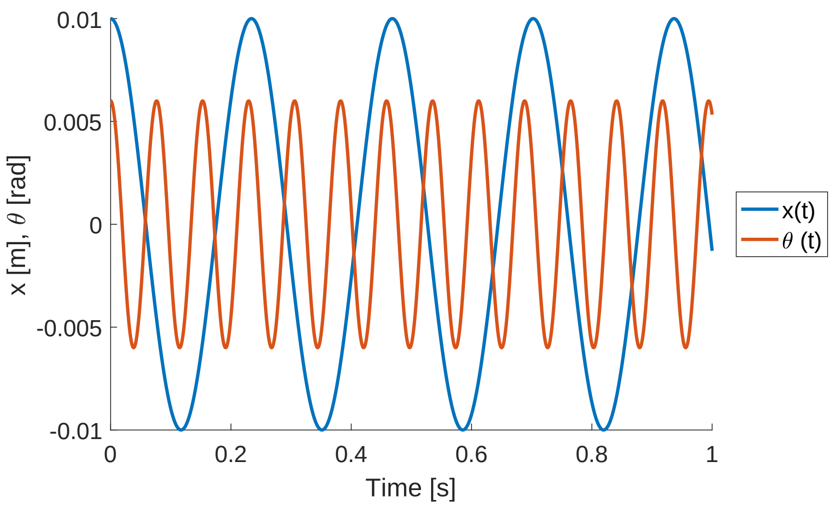

Figure 2: Decoupled motion.

   For $l_2 = l_{2,coupled}$ with $x_0 = 0.01$ m, $\theta_0 = 0.006$ rad, $\dot{x}_0 = 0$ m/s and $\dot{\theta}_0 = 0$ rad/s, we have the following coupled motions:

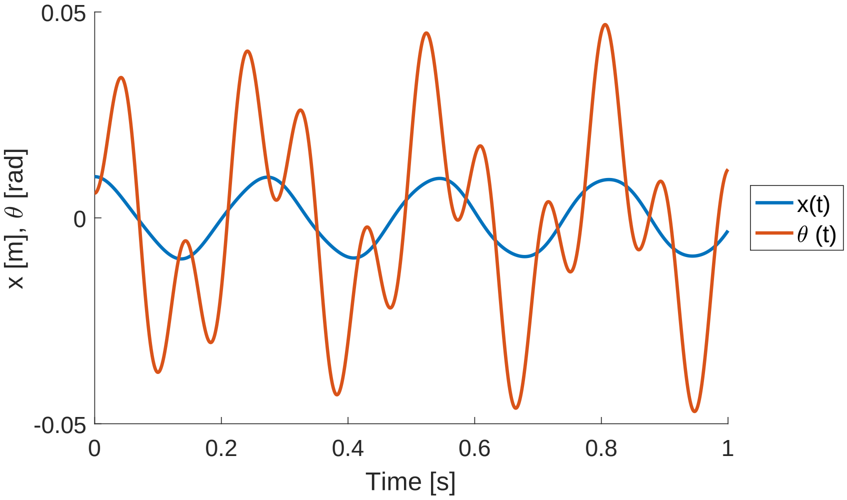

Figure 3: Coupled motion.

2 Tuned Mass Damper

1. The equations of motion of the mechanical system are:

$$\begin{aligned} m_s\ddot{x}_s + c_s\dot{x}_s + k_s x_s + c_d(\dot{x}_s - \dot{x}_d) + k_d(x_s - x_d) = Ae^{i\omega t} \\ m_d\ddot{x}_d + c_d(\dot{x}_d - \dot{x}_s) + k_d(x_d - x_s) = 0 \end{aligned} \tag{7}$$

   Or in matrix form:

$$\mathbf{M}\begin{bmatrix}\ddot{x}_s \\ \ddot{x}_d\end{bmatrix} + \mathbf{C}\begin{bmatrix}\dot{x}_s \\ \dot{x}_d\end{bmatrix} + \mathbf{K}\begin{bmatrix}x_s \\ x_d\end{bmatrix} = \begin{bmatrix}Ae^{i\omega t} \\ 0\end{bmatrix} \tag{8}$$

2. Introducing the harmonic motions of the system in the forms $x_s(t) = X_s e^{i\omega t}$ and $x_d(t) = X_d e^{i\omega t}$, the system 7 can be rewritten in matrix form as:

$$-\omega^2\mathbf{M}\begin{bmatrix}X_s \\ X_d\end{bmatrix} + i\omega\mathbf{C}\begin{bmatrix}X_s \\ X_d\end{bmatrix} + \mathbf{K}\begin{bmatrix}X_s \\ X_d\end{bmatrix} = \begin{bmatrix}A \\ 0\end{bmatrix} \tag{9}$$

   The system 9 can be furthermore simplified as :

$$\begin{bmatrix}X_s \\ X_d\end{bmatrix} = \mathbf{Z}^{-1}\begin{bmatrix}A \\ 0\end{bmatrix} \tag{10}$$

   where $\mathbf{Z}$ is a complex matrix expressed as,

$$\mathbf{Z} = -\omega^2\mathbf{M} + i\omega\mathbf{C} + \mathbf{K} \tag{11}$$

   As such, the displacemnt X_s is given by:

$$X_s(i\omega) = \frac{Z_{22}A}{Z_{11}Z_{22} - Z_{12}^2} \tag{12}$$

X_s is a complex number of the form $z = \frac{a+ib}{c+id}$ which amplitude is given by $|z| = \sqrt{\frac{a^2+b^2}{c^2+d^2}}$. Use Matlab or Python to solve those equations. Figure 4 below show the evolution of $|X_s|$ for values of ω ranging between $0.9\omega_s$ and $1.1\omega_s$ for the case with TMD (c.f. equation 7) and the case without TMD (the mass m_s alone). The effect of the TMD can clearly be seen.

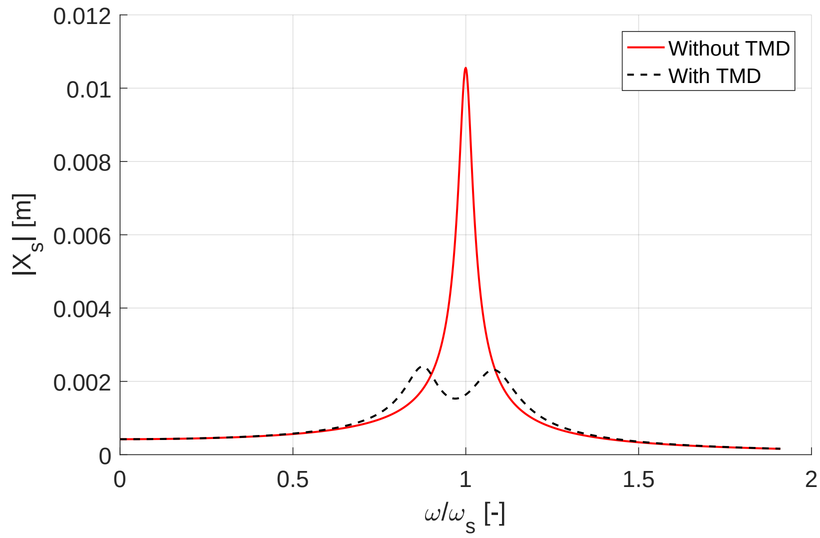

Figure 4: Structure displacement with and without TMD.

3. TMDs stand for tuned mass dampers. Their effectiveness is indeed optimized when the natural frequency of the damping mass is close to the one of the structure. This effect can be exemplified by doubling the mass of the damper (while keeping every other parameter unchanged). Figure 5 clearly shows that the damping properties are more than twice reduced when m_d is modified.

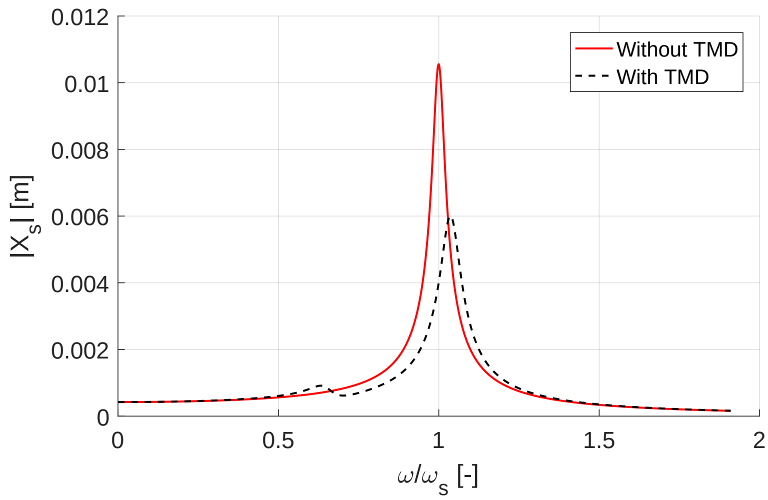

Figure 5: Structure displacement with an untuned TMD and without TMD.