AEROELASTICITY AND FLUID-STRUCTURE INTERACTION

Chapter 5:

Quasi-Static Aeroelasticity: Dynamic Instability - Flutter

Dynamic Instability - Flutter

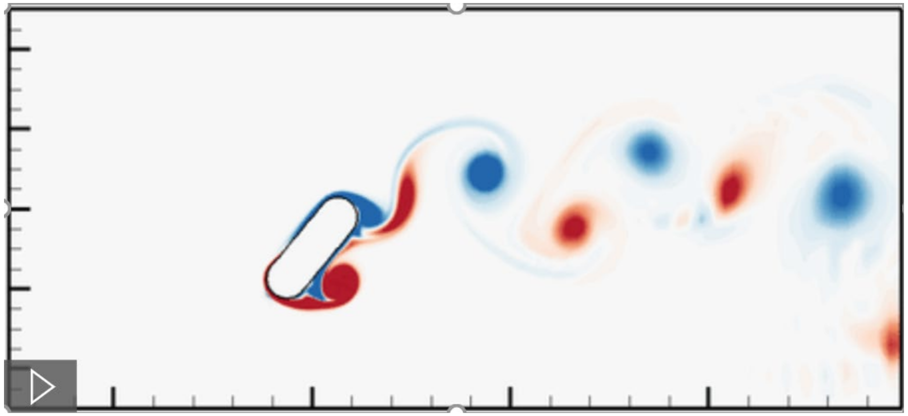
Definition (Cambridge Dictionary):

Flutter: ***To make a series of quick delicate movements up and down or from side to side, or cause something to do this***

- ***Examples:***
  - ***Brightly colored flags were fluttering in the breeze***
  - ***A white bird poised on a wire and fluttered its wings***
  - ***Every time I think about my exams my stomach flutters!***
- ***En Français: Flottement***

In engineering:

"Flutter is an unstable, self-excited structural oscillation at a definite frequency where energy is extracted from the flow by the motion of the structure"



Dynamic Instability - Flutter

Quasi-static aeroelasticity framework :

- ***Large reduced velocity:*** $$U_R = \frac{T_{solid}}{T_{fluid}} \gg 1$$

  - ***Static instability (divergence):***
    - ***Always involves one vibration mode***
    - ***Example: Torsional divergence of an airfoil***

  - ***Dynamic instability :***
    - ***Oscillations with increasing amplitude***
    - ***Always involves two vibration modes (e.g. torsional & bending modes)***

Dynamic Instability - Flutter

Large reduced velocity: $U_R = \frac{T_{solid}}{T_{fluid}} \gg 1$

- **Dynamic instability : Oscillations with increasing amplitude**

Example: Instability induced by torsion and bending modes

airplane stabilizers

Glider wings

Dynamic Instability - Flutter

Large reduced velocity: $\boldsymbol{U_R = \frac{T_{solid}}{T_{fluid}} \gg 1}$

- ***Dynamic instability : Oscillations with increasing amplitude***

Example: Flapping flag
(Flottement de drapeau)

Energy harvesting from air-induced motion of a piezoelectric flag

Reference: Simultaneous wind and solar energy harvesting with inverted flags

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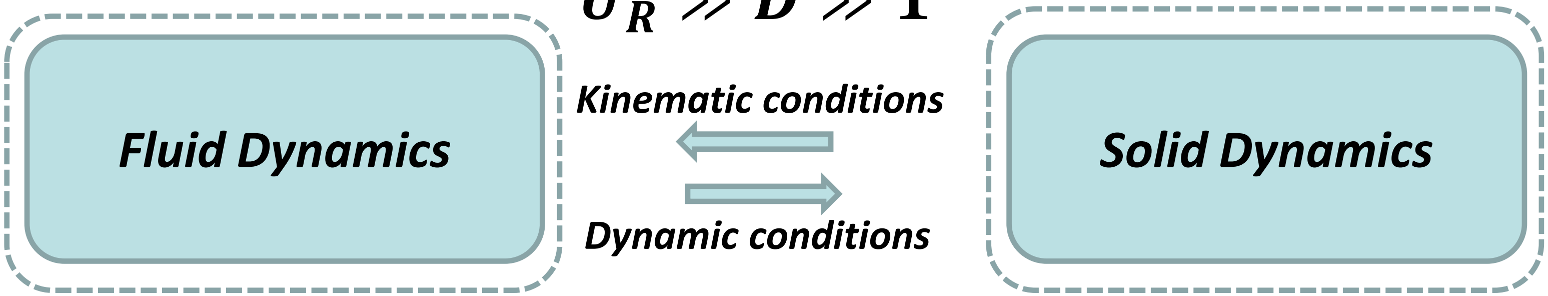
Quasi-Static aeroelasticity – 2 modes approximation

$$U_R \gg D \gg 1$$



Fast motion in the fluid | *Slow motion in the solid*

Hypothesis: The solid motion is the combination of two modes:

$$\xi_s(x,t) = Dq_{1s}(t)\phi_1(x) + Dq_{2s}(t)\,\phi_2(x)$$

Two modal equations of motion:

$$\begin{cases} m_1\ddot{q}_1 + k_1 q_1 = f\phi_1(x) \\ m_2\ddot{q}_2 + k_2 q_2 = f\phi_2(x) \end{cases}$$

$f\phi_1(x)$, $f\phi_2(x)$: *the projections of the fluid loading (the same) on mode shapes ϕ_1 and ϕ_2*

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Quasi-Static aeroelasticity – 2 modes approximation

Steady state fluid dynamics:
Depends on instantaneous position of the interface (q_1 and q_2)

$$p_f\ (Dq_{1s}, Dq_{2s})\ and \quad U_f(Dq_{1s}, Dq_{2s})$$

Projection of the fluid loading on the modes $\phi_1(x)$ and $\phi_2(x)$:

$$f_s\phi_1(x) = C_y F_1(Re, Dq_{1s}, Dq_{2s})$$
$$f_s\phi_2(x) = C_y F_2(Re, Dq_{1s}, Dq_{2s})$$

Quasi-Static aeroelasticity – 2 modes approximation

Remember (Chapter 4):

- ***Single mode approximation (quasi-static aeroelasticity):***

$$\xi_s\,(x_s, t_s) = Dq_s(t_s)\phi(x_s) \qquad p_f\,(Dq_s)\ and\quad U_f(Dq_s)$$

$$Df_s = Cy\int_I \left\{\left[-p_f I + \frac{1}{Re}\left(\nabla U_f + \nabla^t U_f\right)\right].n\right\}.\phi dS = C_y F(Re, Dq_s)$$

- ***Expansion of fluid loading (small displacement Dq_s):***

$$C_y F(Re, Dq_s) = C_y F^0 + C_y\left(\frac{\partial F}{\partial Dq_s}\right)Dq_s + \cdots$$

→ ***The solid behaves like if it was attached to a spring with a stiffness***

$k_f = -C_y\left(\frac{\partial F}{\partial Dq_s}\right)$ ***with a risk of static instability (divergence), when*** $k_f < 0$

A similar approach may be applied to the case of 2 modes

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Quasi-Static aeroelasticity – 2 modes approximation

- ***Expansion of fluid loading (small displacement Dq_{1s} and Dq_{2s}):***
  - ***Flow induced force from the motion of the interface:***

$$f_s\phi_1(x) = C_y{F_1}^0 + C_y\left(\frac{\partial F_1}{\partial Dq_{1s}}\right)Dq_{1s} + C_y\left(\frac{\partial F_1}{\partial Dq_{2s}}\right)Dq_{2s} + \cdots$$

$$f_s\phi_2(x) = C_y{F_2}^0 + C_y\left(\frac{\partial F_2}{\partial Dq_{1s}}\right)Dq_{1s} + C_y\left(\frac{\partial F_2}{\partial Dq_{2s}}\right)Dq_{2s} + \cdots$$

  - ***The terms $C_y{F_1}^0$ and $C_y{F_2}^0$ represent the static part of the loading projections, which are not relevant for the vibration***
  - ***In the following, we will only keep the 1st order terms of the expansion***

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Quasi-Static aeroelasticity – 2 modes approximation

- *Expansion of fluid loading (small displacement Dq_{1s} and Dq_{2s}):*
  - *Flow induced force from the motion of the interface:*

$$\begin{cases} f_s\phi_1(x) = C_y\left(\frac{\partial F_1}{\partial Dq_{1s}}\right)Dq_{1s} + C_y\left(\frac{\partial F_1}{\partial Dq_{2s}}\right)Dq_{2s} + \cdots \\ f_s\phi_2(x) = C_y\left(\frac{\partial F_2}{\partial Dq_{1s}}\right)Dq_{1s} + C_y\left(\frac{\partial F_2}{\partial Dq_{2s}}\right)Dq_{2s} + \cdots \end{cases}$$

$$\Downarrow$$

$$m_1\ddot{q}_1 + k_1 q_1 = C_y\left(\frac{\partial F_1}{\partial q_1}\right)q_1 + C_y\left(\frac{\partial F_1}{\partial q_2}\right)q_2$$

$$m_2\ddot{q}_2 + k_2 q_2 = C_y\left(\frac{\partial F_2}{\partial q_1}\right)q_1 + C_y\left(\frac{\partial F_2}{\partial q_2}\right)q_2$$

Quasi-Static aeroelasticity – 2 modes approximation

- ***Expansion of fluid loading (small displacement Dq_{1s} and Dq_{2s}):***
  - ***Flow induced stiffness (proportional to C_y):***

$$\begin{cases} \ddot{q}_1 + \dfrac{k_1}{m_1} q_1 = \dfrac{C_y}{m_1}\left(\dfrac{\partial F_1}{\partial q_1}\right) q_1 + \dfrac{C_y}{m_1}\left(\dfrac{\partial F_1}{\partial q_2}\right) q_2 \\ \ddot{q}_2 + \dfrac{k_2}{m_2} q_2 = \dfrac{C_y}{m_2}\left(\dfrac{\partial F_2}{\partial q_1}\right) q_1 + \dfrac{C_y}{m_2}\left(\dfrac{\partial F_2}{\partial q_2}\right) q_2 \end{cases}$$

$$\left(\frac{\partial F_i}{\partial q_j}\right) = K_{ij} \qquad i,j = 1,2 \qquad \omega_1 = \sqrt{\frac{k_1}{m_1}} \qquad \omega_2 = \sqrt{\frac{k_2}{m_2}}$$

- ***The 2 modes are coupled with flow-induced stiffness forces***

Quasi-Static aeroelasticity – 2 modes approximation

- *Expansion of fluid loading (small displacement Dq_{1s} and Dq_{2s}):*
  - *Flow induced stiffness (proportional to C_y):*

$$\Rightarrow \begin{cases} \ddot{q}_1 + {\omega_1}^2 q_1 = C_y \dfrac{K_{11}}{m_1} q_1 + C_y \dfrac{K_{12}}{m_1} q_2 \\ \ddot{q}_2 + {\omega_2}^2 q_2 = C_y \dfrac{K_{21}}{m_2} q_1 + C_y \dfrac{K_{22}}{m_2} q_2 \end{cases}$$

K_{11}, K_{22} : *Similar to single mode approximation*

K_{12}, K_{21} : *Coupled stiffness resulting from the interaction between the 2 modes*

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Quasi-Static aeroelasticity – 2 modes approximation

- ***Solid motion under coupled flow-induced stiffness forces:***

$$\begin{cases} \ddot{q}_1 + \left(\omega_1{}^2 - C_y \dfrac{K_{11}}{m_1}\right) q_1 = C_y \dfrac{K_{12}}{m_1} q_2 \\ \ddot{q}_2 + \left(\omega_2{}^2 - C_y \dfrac{K_{22}}{m_2}\right) q_2 = C_y \dfrac{K_{21}}{m_2} q_1 \end{cases}$$

  - ***As the flow velocity is increased, the frequencies of mode 1 and 2 are altered in different ways***
    - ***These frequencies may come close to each other or move away from each other***

Dynamic Instability - Flutter

Quasi-Static aeroelasticity – 2 modes approximation

- ***Solid motion under coupled flow-induced stiffness forces:***
  - ***We consider the case of "coincidence of frequencies":***

$$\omega_1{}^2 - C_y \frac{K_{11}}{m_1} = \omega_2{}^2 - C_y \frac{K_{22}}{m_2}$$

- ***In this case, a common reference time allows writing the equations of solid motion in non dimensional way.***
  Coincidence → Non dimensional frequency =1 for both modes

$$\begin{cases} \ddot{q}_{1s} + q_{1s} = C_y K'_{12} q_{2s} \\ \ddot{q}_{2s} + q_{2s} = C_y K'_{21} q_{1s} \end{cases}$$

K'_{12} ***and*** K'_{21} ***are non-dimensional forms of coupled stiffnesses*** K_{12} ***and*** K_{21}

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Quasi-Static aeroelasticity – 2 modes approximation

- *Solid motion in the case of "coincidence of frequencies":*
  - *Assumptions (for simplicity):*
    - *Coupling stiffness very small with respect of modes stiffness:*
    - *Same magnitude of coupling stiffness forces:*

$$\left|C_y K'_{12}\right| = \left|C_y K'_{21}\right| = \varepsilon \ll 1$$

  - *The equations for the solid motion read:*

$$\begin{cases} \ddot{q}_{1s} + q_{1s} = \varepsilon q_{2s} \\ \ddot{q}_{2s} + q_{2s} = \varepsilon q_{1s} \end{cases} \quad or \quad \begin{cases} \ddot{q}_{1s} + q_{1s} = \varepsilon q_{2s} \\ \ddot{q}_{2s} + q_{2s} = -\varepsilon q_{1s} \end{cases}$$

Symmetric stiffness coupling — *Anti-symmetric stiffness coupling*

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Quasi-Static aeroelasticity – 2 modes approximation

- **Symmetric stiffness coupling:** $$\begin{cases} \ddot{q}_{1s} + q_{1s} = \varepsilon q_{2s} \\ \ddot{q}_{2s} + q_{2s} = \varepsilon q_{1s} \end{cases}$$
  - *We look for harmonic solutions q_{1s} and q_{2s} in the following form:*

$$\begin{pmatrix} q_{1s} \\ q_{2s} \end{pmatrix} = \begin{pmatrix} q_{1s,0} \\ q_{2s,0} \end{pmatrix} e^{i\omega t}$$

$$\Rightarrow \begin{pmatrix} \ddot{q}_{1s} \\ \ddot{q}_{2s} \end{pmatrix} = -\omega^2 \begin{pmatrix} q_{1s} \\ q_{2s} \end{pmatrix}$$

$$\Rightarrow \underbrace{\begin{pmatrix} 1-\omega^2 & -\varepsilon \\ -\varepsilon & 1-\omega^2 \end{pmatrix}}_{\substack{D \\ \text{(Dynamic Matrix)}}} \begin{pmatrix} q_{1s,0} \\ q_{2s,0} \end{pmatrix} e^{i\omega t} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

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Quasi-Static aeroelasticity – 2 modes approximation

- **Symmetric stiffness coupling:**

$$D\begin{pmatrix} q_{1s,0} \\ q_{2s,0} \end{pmatrix} e^{i\omega t} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

- **D stands for the dynamic matrix (not displacement number !).** **For non trivial solutions $(q_{1s}, q_{2s}) \neq (0, 0)$, the determinant of the dynamic matrix must vanish:**

$$Det(D) = 0 \;\Rightarrow\; \left(1-\omega^2\right)^2 = \varepsilon^2 \;\Rightarrow \begin{cases} \omega_A = \sqrt{1+\varepsilon} \approx 1 + \dfrac{\varepsilon}{2} \\ \omega_B = \sqrt{1-\varepsilon} \approx 1 - \dfrac{\varepsilon}{2} \end{cases}$$

$\varepsilon \ll 1 \;\Rightarrow$ ***The new frequencies are real positive numbers***

Dynamic Instability - Flutter

Quasi-Static aeroelasticity – 2 modes approximation

- ***Symmetric stiffness coupling**:*

→ *The eigenvectors may be obtained for ω_A and ω_B by solving:*

$$\begin{pmatrix} 1-\omega_A{}^2 & -\varepsilon \\ -\varepsilon & 1-\omega_A{}^2 \end{pmatrix}\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} 1-\omega_B{}^2 & -\varepsilon \\ -\varepsilon & 1-\omega_B{}^2 \end{pmatrix}\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

By using : $1-\omega_A{}^2 = -\varepsilon \;\; ; \;\; 1-\omega_B{}^2 = \varepsilon$ *and by setting* $q_1 = 1$*,*
we obtain 2 eigenvectors A and B:

$$Q_A = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}_A = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad Q_B = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}_B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

→ *In presence of symmetric coupled stiffness, the new modes A and B are combinations of original modes 1 and 2 with slightly altered frequencies (weak coupling)*

Dynamic Instability - Flutter

Quasi-Static aeroelasticity – 2 modes approximation

- **Symmetric stiffness coupling:**
  - **Modal masses**

Transpose

$$M_A = Q_A{}^t \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} Q_A = 2 \qquad M_B = Q_B{}^t \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} Q_B = 2$$

  - **Modal stiffnesses**

$$K_A = Q_A{}^t \begin{pmatrix} 1-\omega^2 & -\varepsilon \\ -\varepsilon & 1-\omega^2 \end{pmatrix} Q_A = 2 + 2\varepsilon$$

$$K_B = Q_B{}^t \begin{pmatrix} 1-\omega^2 & -\varepsilon \\ -\varepsilon & 1-\omega^2 \end{pmatrix} Q_B = 2 - 2\varepsilon$$

We verify that :

$$\omega_A{}^2 = \frac{K_A}{M_A} = 1 + \varepsilon \quad and \quad \omega_B{}^2 = \frac{K_B}{M_B} = 1 - \varepsilon$$

Dynamic Instability - Flutter

Quasi-Static aeroelasticity – 2 modes approximation

- *Anti-symmetric stiffness coupling*: $\begin{cases} \ddot{q}_{1s} + q_{1s} = \varepsilon q_{2s} \\ \ddot{q}_{2s} + q_{2s} = -\varepsilon q_{1s} \end{cases}$
  - *We look for solutions q_{1s} and q_{2s} in the following forms:*

$$\begin{pmatrix} q_{1s} \\ q_{2s} \end{pmatrix} = \begin{pmatrix} q_{1s,0} \\ q_{2s,0} \end{pmatrix} e^{i\omega t}$$

$$\Rightarrow \begin{pmatrix} 1-\omega^2 & -\varepsilon \\ \varepsilon & 1-\omega^2 \end{pmatrix} \begin{pmatrix} q_{1s,0} \\ q_{2s,0} \end{pmatrix} e^{i\omega t} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

The determinant of the dynamic matrix must vanish:

$$\Rightarrow \quad \left(1-\omega^2\right)^2 = -\varepsilon^2 \quad \Rightarrow \quad \begin{cases} \omega_A = \sqrt{1+i\varepsilon} \approx 1 + i\dfrac{\varepsilon}{2} \\ \omega_B = \sqrt{1-i\varepsilon} \approx 1 - i\dfrac{\varepsilon}{2} \end{cases}$$

The new frequencies are complexe numbers

Dynamic Instability - Flutter

Quasi-Static aeroelasticity – 2 modes approximation

- **Anti-symmetric stiffness coupling: Solid motion**

→ *We obtain 2 modes A and B with complex frequencies and complex eigenvectors (as we did for symmetric case):*

$$\begin{cases} \omega_A = 1 + i\dfrac{\varepsilon}{2} \\ Q_A = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}_A = \begin{pmatrix} 1 \\ -i \end{pmatrix} \end{cases} \quad \text{and} \quad \begin{cases} \omega_B = 1 - i\dfrac{\varepsilon}{2} \\ Q_B \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}_B = \begin{pmatrix} 1 \\ i \end{pmatrix} \end{cases}$$

Dynamic Instability - Flutter

Quasi-Static aeroelasticity – 2 modes approximation

- *Anti-symmetric stiffness coupling*: *Solid motion*
  - *Modal masses and stiffness*

$$M_A = Q_A{}^* \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} Q_A = 2 \qquad M_B = Q_B{}^* \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} Q_B = 2$$

Conjugate

$$K_A = Q_A{}^* \begin{pmatrix} 1-\omega^2 & -\varepsilon \\ \varepsilon & 1-\omega^2 \end{pmatrix} Q_A = 2 + 2i\varepsilon$$

$$K_B = Q_B{}^* \begin{pmatrix} 1-\omega^2 & -\varepsilon \\ \varepsilon & 1-\omega^2 \end{pmatrix} Q_B = 2 - 2i\varepsilon$$

We verify that :

$$\omega_A{}^2 = \frac{K_A}{M_A} = 1 + i\varepsilon \quad \textit{and} \quad \omega_B{}^2 = \frac{K_B}{M_B} = 1 - i\varepsilon$$

Quasi-Static aeroelasticity – 2 modes approximation

- **Anti-symmetric stiffness coupling: Solid motion**
  - ***The first mode A:***

$$\begin{cases} \omega_A = 1 + i\dfrac{\varepsilon}{2} \\ \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}_A = \begin{pmatrix} 1 \\ -i \end{pmatrix} \end{cases}$$

$$\begin{pmatrix} q_{1s} \\ q_{2s} \end{pmatrix} = Re\left[\begin{pmatrix} q_1 \\ q_2 \end{pmatrix}_A e^{i\omega_A t}\right] = Re\left[\begin{pmatrix} 1 \\ -i \end{pmatrix} e^{i\left(1+i\frac{\epsilon}{2}\right)t}\right]$$

$$\begin{pmatrix} q_{1s} \\ q_{2s} \end{pmatrix} = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} e^{-\frac{\epsilon t}{2}}$$

→ ***The mode A is a damped mode with the vibration amplitude vanishing in exponential way with time***

Dynamic Instability - Flutter

Quasi-Static aeroelasticity – 2 modes approximation

- ***Anti-symmetric stiffness coupling**: Solid motion*
  - ***The second mode B:***

$$\begin{cases} \omega_B = 1 - i\dfrac{\varepsilon}{2} \\ \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}_B = \begin{pmatrix} 1 \\ i \end{pmatrix} \end{cases}$$

$$\begin{pmatrix} q_{1s} \\ q_{2s} \end{pmatrix} = Re\left[\begin{pmatrix} q_1 \\ q_2 \end{pmatrix}_B e^{i\omega_B t}\right] = Re\left[\begin{pmatrix} 1 \\ i \end{pmatrix} e^{i\left(1 - i\frac{\epsilon}{2}\right)t}\right]$$

$$\begin{pmatrix} q_{1s} \\ q_{2s} \end{pmatrix} = \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} e^{\frac{\epsilon t}{2}}$$

→ ***The second mode B is an unstable mode with the vibration amplitude increasing exponentially without a limit***

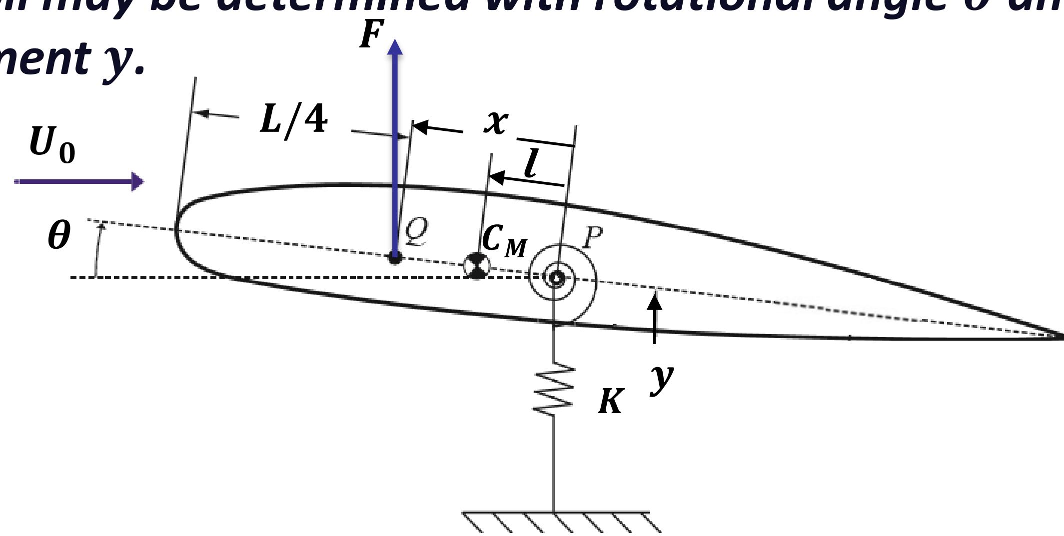
→ ***Dynamic instability or Flutter***

Quasi-Static aeroelasticity – 2 modes approximation

- ***Dynamic Instability – Summary:***
  - *In presence of coincidence of frequencies of 2 vibration modes :*
    - *Symmetric stiffness coupling leads to new modes (combination of original modes) with a slightly altered frequencies*
      This is a conservative coupling with no energy transfer between the modes within one cycle
    - *Anti-symmetric stiffness coupling leads to new modes with complex frequencies and eigenvectors: a damped mode and an unstable mode*
      This unstable mode is non conservative: Energy is exchanged between the two modes and accumulates in time.
      This instability is called dynamic instability or flutter
  - *Quasi-static elasticity*
    → Prediction of static & dynamic instabilities (Divergence & flutter)

Example: 2 modes approximation of an airfoil

- We consider a 2D airfoil with torsion and plunge modes:
  - Attached at P to torsional and translational springs with stiffness C and K:
    - L : Chord length
    - C_M: center of mass, located at a distance l from elastic center (l positive towards the leading edge)
    - Q: aerodynamic center, located at a distance x from elastic center (x positive towards the leading edge)
    - The position of the foil may be determined with rotational angle θ and the vertical displacement y.



Dynamic Instability - Flutter

2 modes approximation of an airfoil

- **In absence of flow (U_0=0)**

→ *Classical problem of solid mechanics. May be solved with Lagrange equation*

- **Kinetic energy:** $$E_c = \frac{1}{2}M(\dot{y} + l\dot{\theta})^2 + \frac{1}{2}J\dot{\theta}^2$$

M: Mass
J: Moment of inertia

- **Potential energy (Elastic energy):** $$E_p = \frac{1}{2}Ky^2 + \frac{1}{2}C\theta^2$$

- **Lagrangian:** $$\mathcal{L} = E_c - E_p$$

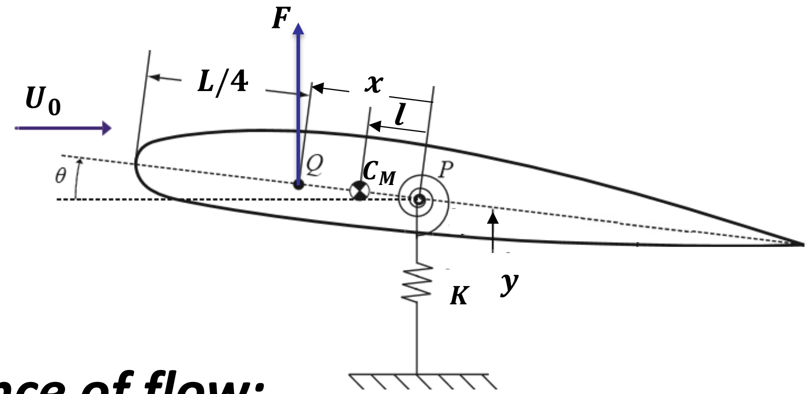
$$\begin{cases} \dfrac{d}{dt}\left(\dfrac{\partial \mathcal{L}}{\partial \dot{y}}\right) - \dfrac{\partial \mathcal{L}}{\partial y} = 0 \\ \dfrac{d}{dt}\left(\dfrac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) - \dfrac{\partial \mathcal{L}}{\partial \theta} = 0 \end{cases} \Rightarrow \begin{cases} M\ddot{y} + Ml\ddot{\theta} + Ky = 0 \\ J\ddot{\theta} + C\theta + (M\ddot{y} + Ml\ddot{\theta})l = 0 \end{cases}$$

$$\Rightarrow \begin{cases} M\ddot{y} + Ml\ddot{\theta} + Ky = 0 \\ J\ddot{\theta} + C\theta - Kyl = 0 \end{cases}$$

Dynamic Instability - Flutter

2 modes approximation of an airfoil

- *In presence of a flow and with large reduced velocity* $(UR \gg 1)$
  - *The fluid force acting on the surface of the airfoil, at any frozen position defined by θ and y:*



$$F = \frac{1}{2}\rho U_0^2 L C_L(\theta)$$ L: Chord length

  - *Equations of solid motion in presence of flow:*

$$\begin{cases} M\ddot{y} + Ml\ddot{\theta} + Ky = \frac{1}{2}\rho U_0^2 L C_L(\theta) \\ J\ddot{\theta} + C\theta - Kyl = +\frac{1}{2}\rho U_0^2 L(x-l) C_L(\theta) \end{cases}$$

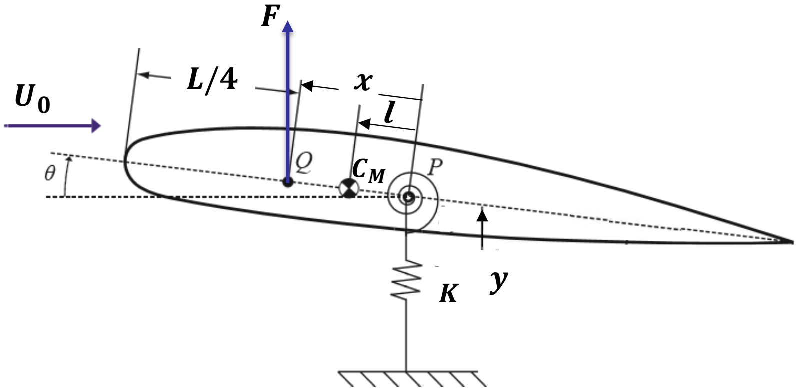
Dynamic Instability - Flutter

2 modes approximation of an airfoil

- **Thin airfoil hypothesis:**

$$C_L = 2\pi\theta \quad \Rightarrow \quad F = \frac{\rho U_0^2}{2}(2\pi\theta)\mathrm{L} = \pi\rho U_0^2\theta\mathrm{L}$$

- **Dimesionless variables:**



$$q_{1s} = \frac{y}{L}; \quad q_{2s} = \theta; \quad \epsilon = \frac{l}{L}; \quad x_s = \frac{x}{L}$$

$$\Omega = \sqrt{\frac{KJ}{CM}}$$: Ratio of translational and rotational frequencies

$$\kappa = \frac{KL^2}{C}$$: Stiffness ratio

$$Cy = \frac{\rho {U_0}^2 L^2}{2C}$$: Cauchy Number

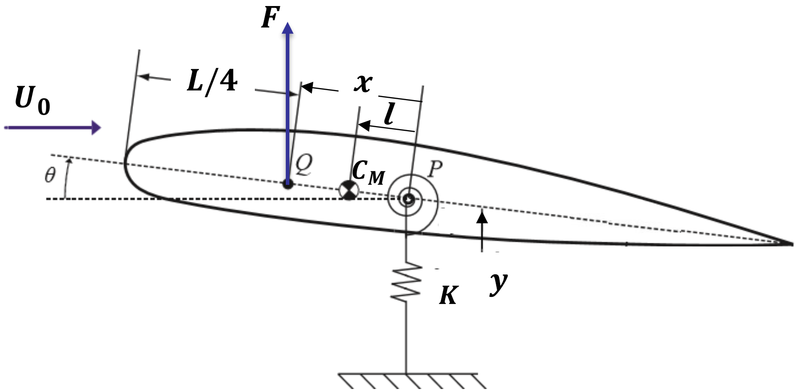
$$t_s = \sqrt{\frac{C}{J}}t$$: dimensionless time, scaled by torsional period

Dynamic Instability - Flutter

2 modes approximation of an airfoil

- *Dimensionless form of coupled equations of solid motion in presence of flow:*

$$
\begin{cases}
\ddot{q}_{1s} = \dfrac{d^2 q_{1s}}{dt_s{}^2} = \dfrac{J\ddot{y}}{LC} \\
\ddot{q}_{2s} = \dfrac{d^2 q_{2s}}{dt_s{}^2} = \dfrac{J\ddot{\theta}}{C}
\end{cases}
$$



$$
\Rightarrow \quad
\begin{cases}
\ddot{q}_{1s} + \epsilon \ddot{q}_{2s} + \Omega^2 q_{1s} = 2\pi C_Y \dfrac{\Omega^2}{\kappa} q_{2s} \\
\ddot{q}_{2s} + \big(1 - 2\pi C_Y \left(x_s - \epsilon\right)\big) q_{2s} = +\kappa \epsilon q_{1s}
\end{cases}
$$

$x_s - \epsilon > 0 \implies$ *The torsional frequency decreases with increasing C_Y*

Dynamic Instability - Flutter

2 modes approximation of an airfoil

- *We look for harmonic solutions in the following form:*

$$\begin{pmatrix} q_{1s} \\ q_{2s} \end{pmatrix} = Re\left[\begin{pmatrix} q_{1s} \\ q_{2s} \end{pmatrix}_0 e^{i\omega t}\right]$$

$\Rightarrow$ **2 modes:** $\begin{cases} \text{Frequencies: } \omega_A, \omega_B \\ \text{Modal vectors: } \begin{pmatrix} q_{1s} \\ q_{2s} \end{pmatrix}_A, \begin{pmatrix} q_{1s} \\ q_{2s} \end{pmatrix}_B \end{cases}$

$$\Rightarrow \begin{pmatrix} \Omega^2 & -2\pi C_Y \dfrac{\Omega^2}{\kappa} \\ -\kappa\epsilon & 1 - 2\pi C_Y (x_s - \epsilon) \end{pmatrix} \begin{pmatrix} q_{1s} \\ q_{2s} \end{pmatrix} = \omega^2 \begin{pmatrix} 1 & \epsilon \\ 0 & 1 \end{pmatrix} \begin{pmatrix} q_{1s} \\ q_{2s} \end{pmatrix}$$

- *If all the parameters are fixed except the Cauchy number, we may solve numerically the linear system to derive the two roots :*

$$\omega_A(C_Y) = \omega_{A,real}(C_Y) + i\omega_{A,im}(C_Y)$$
$$\omega_B(C_Y) = \omega_{B,real}(C_Y) + i\omega_{B,im}(C_Y)$$

Dynamic Instability - Flutter

2 modes approximation of an airfoil

- ***Coupled equations of solid motion in presence of flow:***
  - ***By increasing the Cauchy number, the complex frequencies may be used to predict the dynamic instability of the system:***
  - ***Dynamic instability occurs when:***

(1)

$$\boldsymbol{\omega_{A,im}\,(C_Y) < 0}$$

or

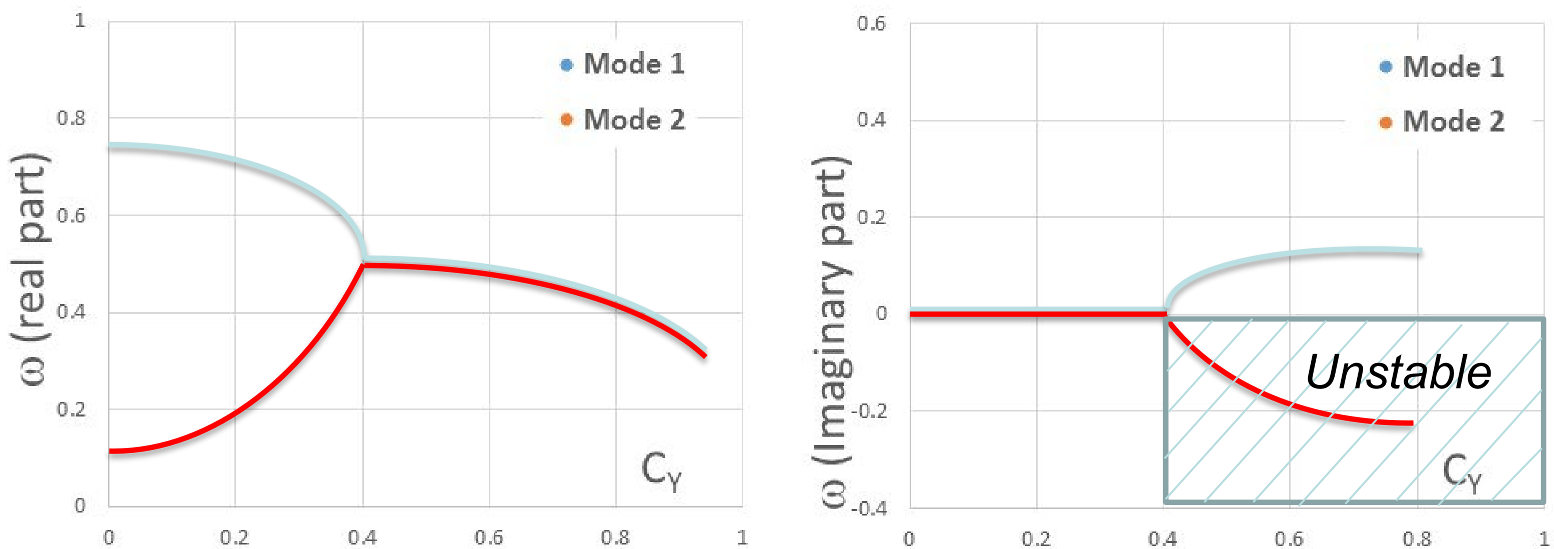
$$\boldsymbol{\omega_{B,\mathrm{im}}\,(C_Y) < 0}$$

Anti-symmetric stiffness coupling (amplitude increases like e^t)

Dynamic Instability - Flutter

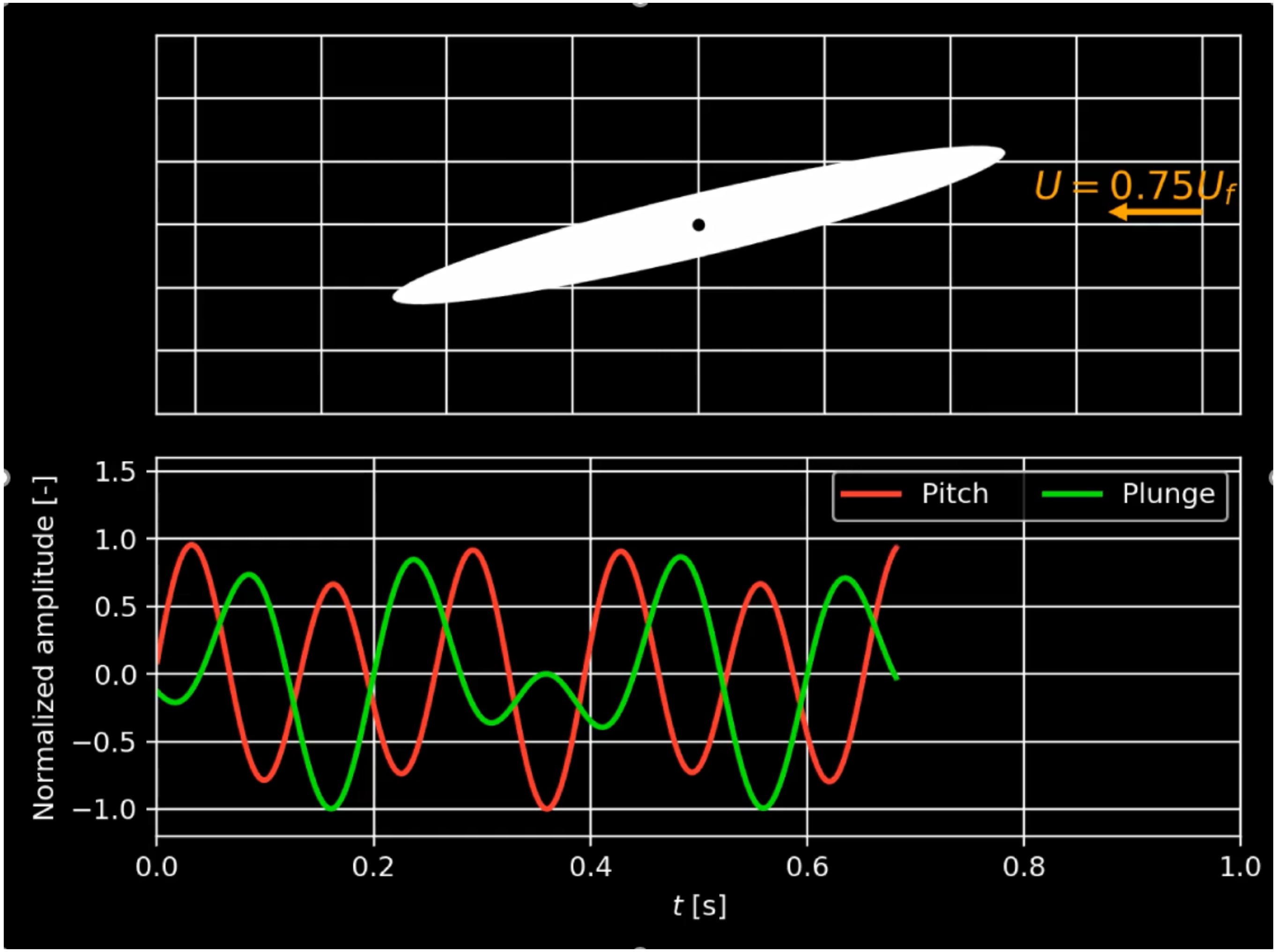
2 modes approximation of an airfoil

- ***Procedure: Plot the real and imaginary parts of the frequencies as functions of Cauchy number and check the condition (1)***
- ***Example of frequency evolution for a given set of parameters:***



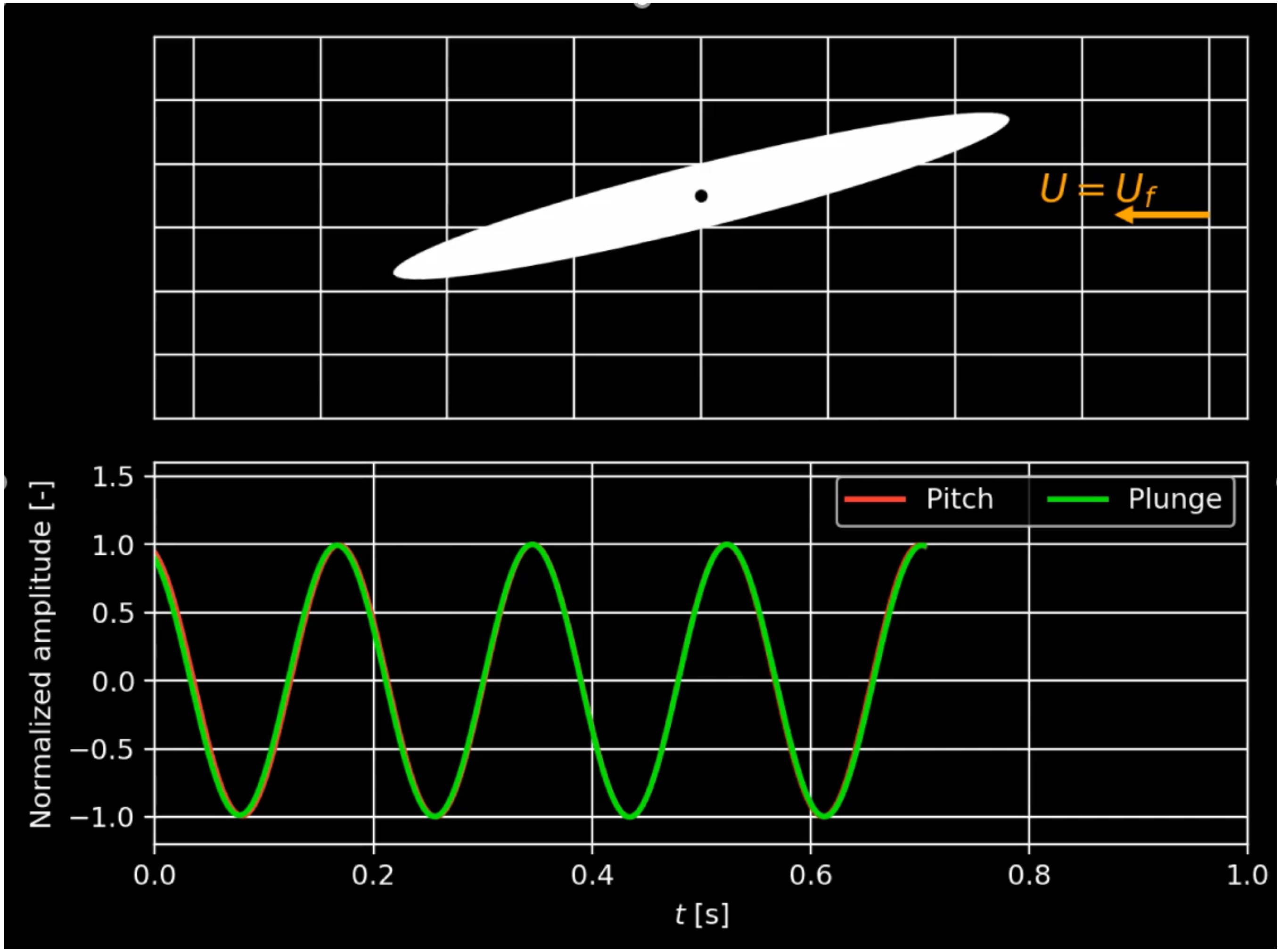
- ***$C_Y < 0.4$ → 2 modes with real values of frequencies (weak coupling)***
- ***$C_Y > 0.4$ → Mode 2 has a frequency with negative imaginary part***
  → Dynamic instability

Numerical solution: Below flutter speed



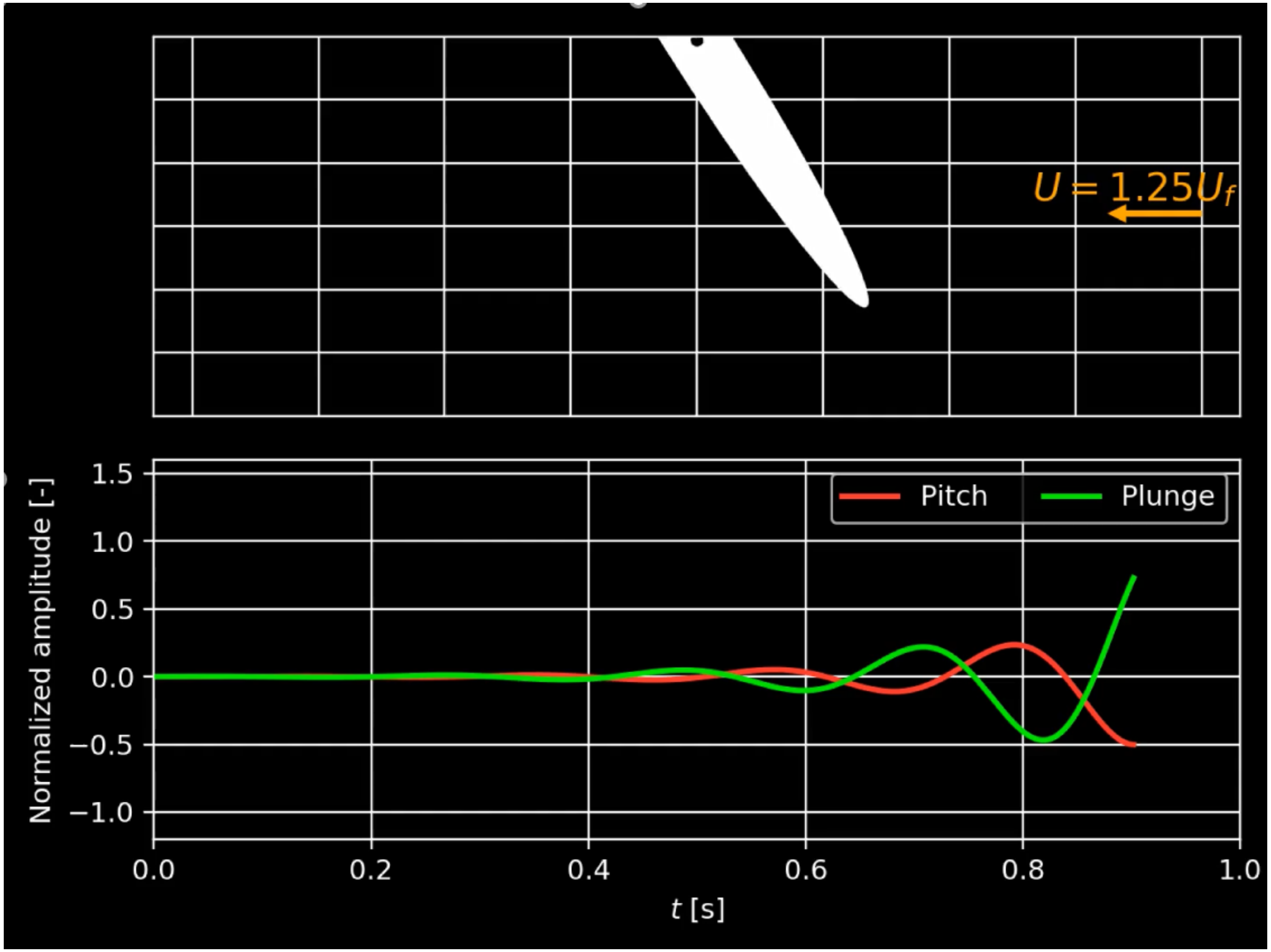
Dynamic Instability - Flutter

Numerical solution: At flutter speed



Dynamic Instability - Flutter

Numerical solution: Beyond flutter speed



Summary

- ***We have seen how quasi-static elasticity framework allows to predict dynamic instability (flutter) involving 2 vibration modes of a foil in a flow***

- ***Remarks:***
  - ***The hypothesis of $U_R \gg 1$ is not always true. What happens at $U_R \approx 1$?***
    - ***In the case of an airfoil, at $U_R \approx 1$, the motion of the solid may not be neglected and may have significant impacts:***
      - ***Effective incidence angle depends on the foil speed → damping effect***
      - ***The fast motion of the foil generates vortices at LE and TE***

  - ***The flutter is not limited to 2 modes, we do see instabilities involving one mode only. This is not covered within the quasi-static aeroelasticity***