AEROELASTICITY AND FLUID-STRUCTURE INTERACTION

Supplementary Material

Mechanical Vibrations, A Reminder

Basic Concepts of Vibration

- ***What are vibrations?***
  - *Any motion that repeats itself after an interval of time (a.k.a. oscillations)*

- ***Elementary parts of a vibrating system:***
  - *Means for storing kinetic energy (mass or inertia)*
  - *Means for storing potential energy (spring or elasticity)*
  - *Means for dissipating energy (damper)*

- ***Number of degree of freedom:***
  - *The minimum number of coordinates required to completely determine all positions of all parts of the system*

- ***Discrete v.s. continuous systems:***
  - *Most systems have an infinite number of degree of freedom → continuous system*
  - *However, continuous systems are often approximated by discrete systems to obtain solutions in a simpler manner → treated as finite lumped masses, springs and dampers*

Basic Concepts of Vibration

Classification of Vibration

Free or Forced

Is the system subjected to an external force?

Undamped or Damped

Does the system dissipate energy as it oscillates?

Linear or Nonlinear

Do the components (spring, damper, …) of the system behave linearly?

Deterministic or Random

Is the value of the external force known at any given time?

Vibrating System Modeling

From a complex vibrating system to a simplified model:

- *Intended to represent all the **important features** of the system in order to derive the equations governing the **system behavior**.*

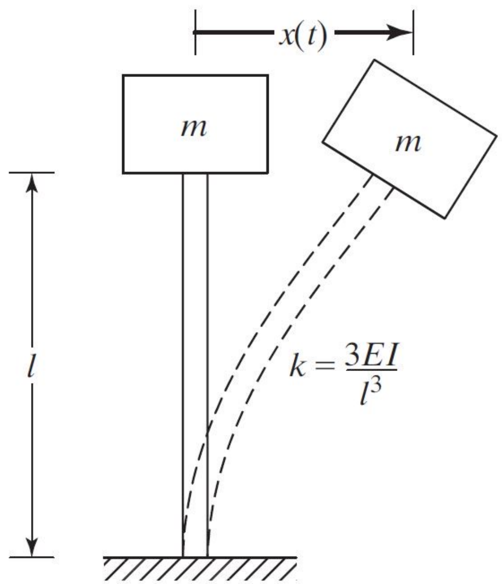
- *Elementary models can give a quick insight into the behavior of a model → but the model can be further refined to obtain more accurate results.*

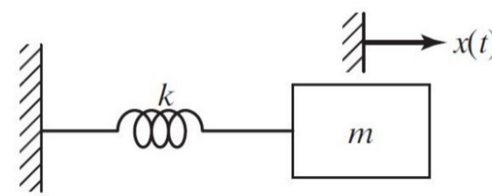
→ *Let's start by the basis: the **single-degree-of-freedom** system or SDOF.*

Single Degree of Freedom Systems

Free Vibration of undamped SDOF systems

- *The system oscillates due to an **initial disturbance** only.*
- *An example: the transverse vibration of a tall structure:*





Actual structure *Simplified model* *Equivalent spring-mass system*

Single Degree of Freedom Systems

Free Vibration of undamped SDOF systems

- *Deriving the equation of motion (Newton second law of motion, conservation of energy, Lagrange's energy equation, …):*

$$m\ddot{x} + kx = 0$$

- *The solution can be found assuming:* $x(t) = Ce^{st}$*, where* C *and* s *are complex numbers to be determined. Injecting in the equation of motion:*

$$C(ms^2 + k) = 0 \quad \Leftrightarrow \quad ms^2 + k = 0$$

$$\Rightarrow s = \pm\left(-\frac{k}{m}\right)^{0.5} = \pm i\omega_n$$

Where $\omega_n = \sqrt{\frac{k}{m}}$ *is the natural frequency of the system.*

→ *Hence the general solution reads:* $x(t) = C_1 e^{i\omega_n t} + C_2 e^{-i\omega_n t}$

Single Degree of Freedom Systems

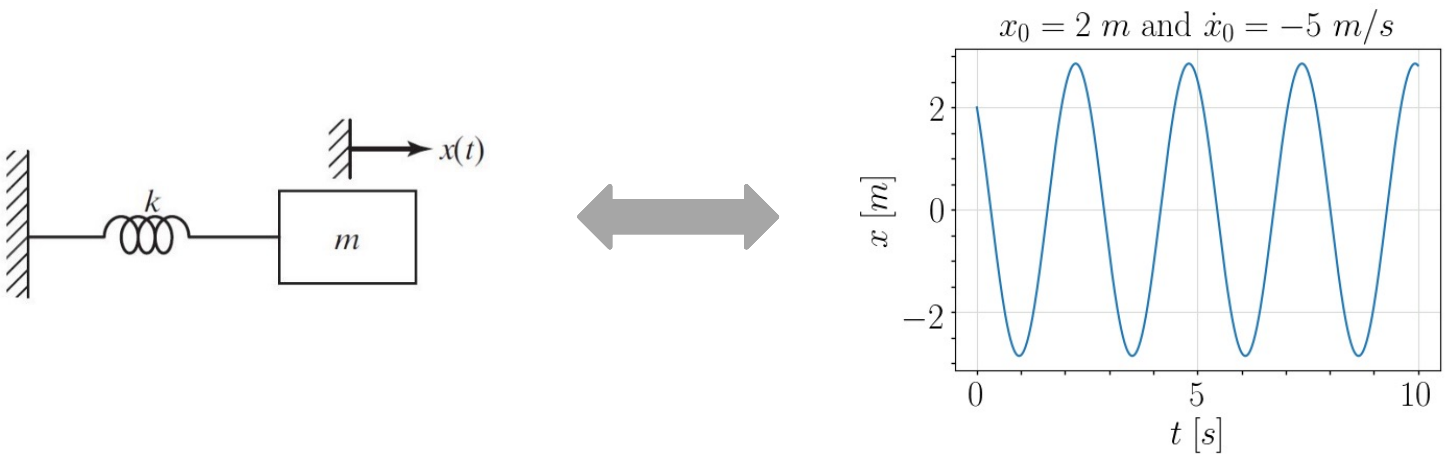
Free Vibration of undamped SDOF systems

- *Considering the real part of the response $x(t)$ only, it can be shown that the solution is given by:*

$$x(t) = A_1 \cos(\omega_n t) + A_2 \sin(\omega_n t) = C \cos(\omega_n t - \phi)$$

Where A_1, A_2, C and ϕ are real constants determined by the initial conditions.

$$\Rightarrow x(t) = x_0 cos(\omega_n t) + \frac{\dot{x}_0}{\omega_n} sin(\omega_n t) = \sqrt{x_0^2 + \left(\frac{\dot{x}_0}{\omega_n}\right)^2} \cos\left(\omega_n t - \tan^{-1}\left(\frac{\dot{x}_0}{x_0 \omega_n}\right)\right)$$



Single Degree of Freedom Systems

Adding damping to freely vibrating SDOF systems

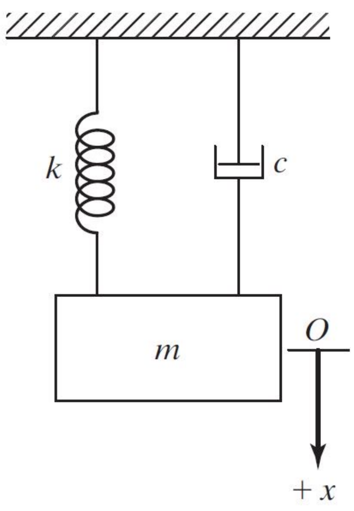
- *The **viscous damping force** is proportional to the **velocity** of the system and opposes the motion. Hence, the equation of motion reads:*

$$m\ddot{x} + c\dot{x} + kx = 0$$

- *The solution can be found assuming: $x(t) = Ce^{st}$, where C and s are to be determined*

$$C(ms^2 + cs + k) = 0 \iff ms^2 + cs + k = 0$$

$$\Rightarrow s_{1,2} = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}$$



→ *Hence the general solution reads: $x(t) = C_1 e^{s_1 t} + C_2 e^{s_2 t}$*

- *A couple notations: critical damping, c_c, and damping ratio ζ*

$$c_c = 2m\omega_n \quad \text{and} \quad \zeta = \frac{c}{c_c}$$

Adding damping to freely vibrating SDOF systems

- *The roots and the solution can thus be rewritten as*

$$s_{1,2} = \left(-\zeta \pm \sqrt{\zeta^2 - 1}\right)\omega_n$$

$$x(t) = C_1 e^{\left(-\zeta+\sqrt{\zeta^2-1}\right)\omega_n t} + C_2 e^{\left(-\zeta-\sqrt{\zeta^2-1}\right)\omega_n t}$$

→ *The behavior of the system hence depends upon the magnitude of the damping.*

- *Three damping conditions exist (excluding undamped vibrations, $\zeta = 0$)*

  - *Underdamped ($\zeta < 1$ or $c < c_c$) → s_1 and s_2 are complex*

$$s_{1,2} = \left(-\zeta \pm i\sqrt{1 - \zeta^2}\right)\omega_n$$

  - *Critically damped ($\zeta = 1$ or $c = c_c$) → s_1 and s_2 are equal*

$$s_1 = s_2 = -\omega_n$$

  - *Overdamped ($\zeta > 1$ or $c > c_c$) → s_1 and s_2 are real*

$$s_{1,2} = \left(-\zeta \pm \sqrt{\zeta^2 - 1}\right)\omega_n < 0$$

Adding damping to freely vibrating SDOF systems.

- *Three damping conditions exist (excluding undamped vibrations, $\zeta = 0$)*

  - *Underdamped:*

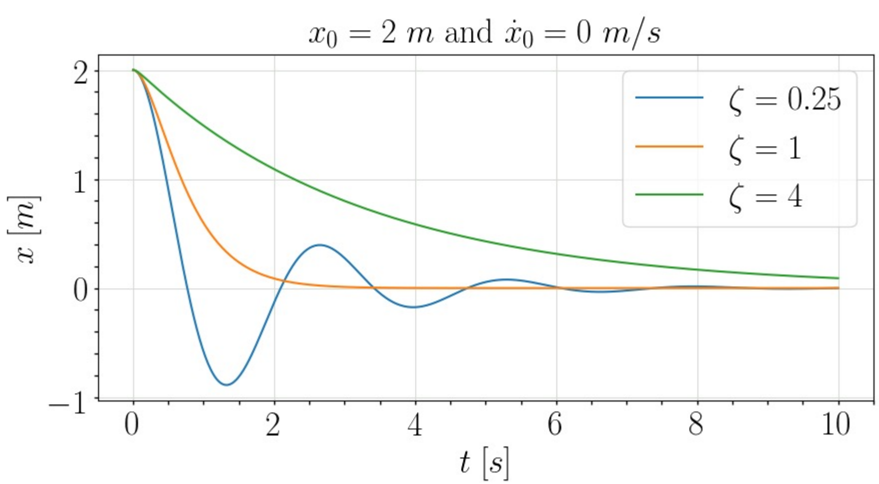
$$x(t) = e^{-\zeta\omega_n t}\left(x_0\cos(\omega_d t) + \frac{\dot{x}_0 + \zeta\omega_n x_0}{\omega_d}\sin(\omega_d t)\right), \qquad \omega_d = \omega_n\sqrt{1-\zeta^2}$$

  - *Critically damped:*

$$x(t) = e^{-\omega_n t}[x_0 + (\dot{x}_0 + \omega_n x_0)t]$$

  - *Overdamped:*

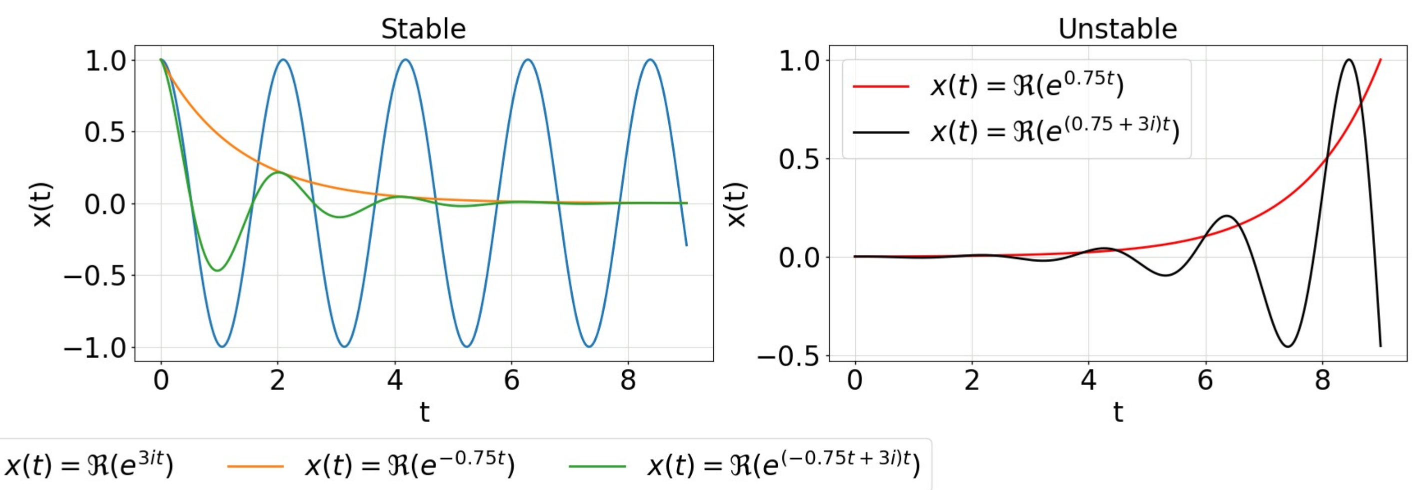
$$x(t) = e^{-\zeta\omega_n t}\left(x_0\cosh(\omega_* t) + \frac{\dot{x}_0 + \zeta\omega_n x_0}{\omega_*}\sinh(\omega_* t)\right), \qquad \omega_* = \omega_n\sqrt{\zeta^2-1}$$



Single Degree of Freedom Systems

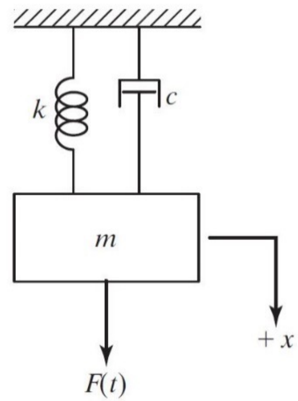
A note on the nature of the solution

- *The solution $x(t) = C_1 e^{s_1 t} + C_2 e^{s_2 t}$ presents the following properties:*
  - *If $s_{1,2}$ has a zero or negative real value, the corresponding response will be **stable**.*
  - *If $s_{1,2}$ has a zero imaginary value, the corresponding response will not oscillate.*
  - *If $s_{1,2}$ has a positive real value, the corresponding response will grow exponentially and will be **unstable**:*
    - *Divergence instability → $s_{1,2}$ has a zero imaginary*
    - *Flutter instability → $s_{1,2}$ has a non-zero imaginary*



Single Degree of Freedom Systems

SDOF system under harmonic forcing: $\boldsymbol{F(t) = F_0 e^{i\omega t}}$



- *The equation of motion reads:*

$$m\ddot{x} + c\dot{x} + kx = F_0 e^{i\omega t}$$

→ *Non-homogeneous equation whose solution is given by the sum of the homogenous (transient) solution, $x_h(t)$, and the particular (steady-state) solution, $x_p(t)$.*

- *By assuming a particular solution of the form $x_p(t) = Xe^{i(\omega t-\phi)} = Xe^{-i\phi}e^{i\omega t}$, we get for the amplitude:*

$$X = \frac{F_0}{[(k-m\omega^2)^2 + c^2\omega^2]^{0.5}} \quad \Leftrightarrow \quad \frac{X}{\delta_{st}} = \frac{1}{[(1-r^2)^2 + (2r\zeta)^2]^{0.5}}$$

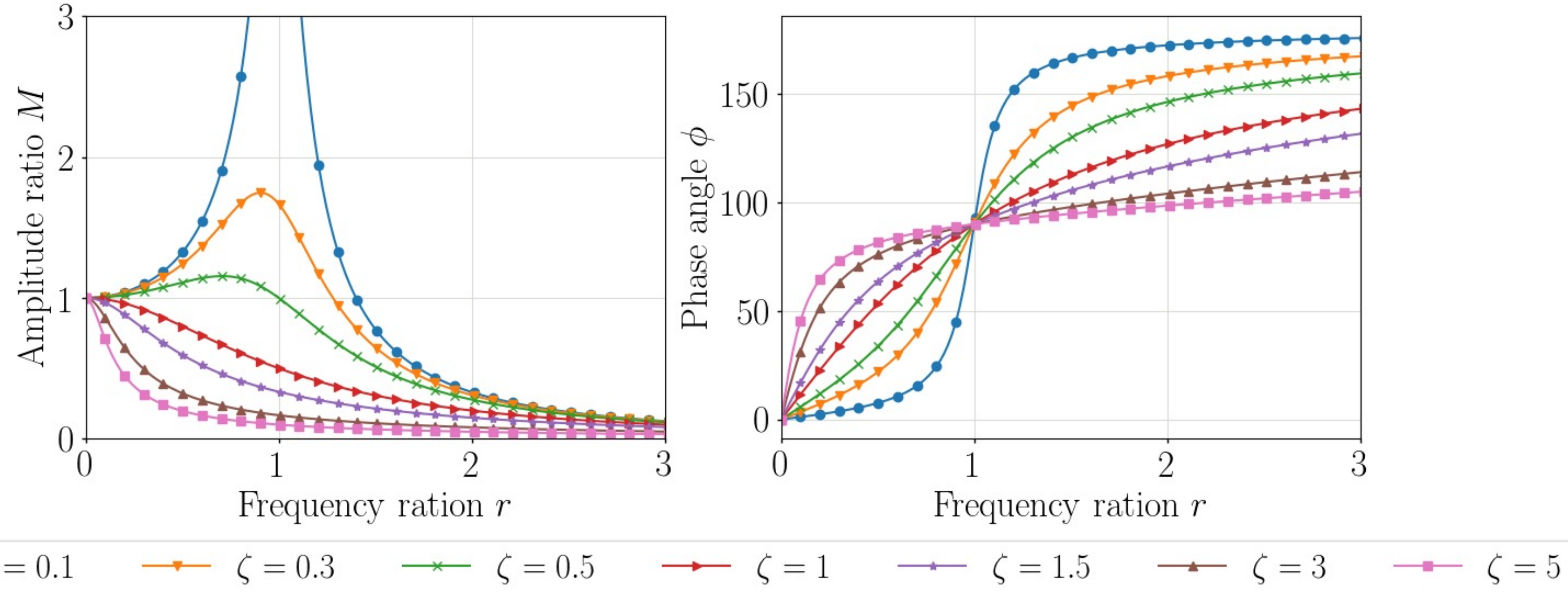
With:

- $\zeta = c/c_c$ *is the damping ratio*
- $\delta_{st} = F_0/k$ *is the deflection under the static force* F_0
- $r = \omega/\omega_n$ *is the frequency ratio*
- $\phi = \tan^{-1}\left(\frac{2\zeta r}{1-r^2}\right)$ *is the phase angle*

Single Degree of Freedom Systems

SDOF system under harmonic forcing: $F(t) = F_0 e^{i\omega t}$

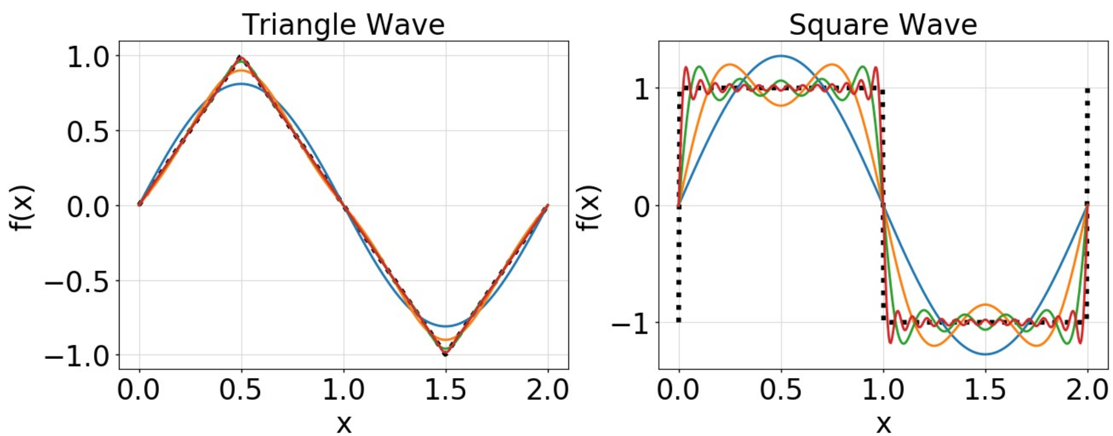
- *Variations of the amplitude ratio, $\frac{X}{\delta_{st}}$, and phase angle as a function of the frequency ratio and damping ratio.*



Single Degree of Freedom Systems

SDOF system under general forcing: a very short overview

- ***Periodic but non-harmonic forcing***:
  - *The periodic forcing can be approximated by a sum of harmonic functions (Fourier series)*
  - *The system response is found by superposing the responses of the indivudal harmonic forcing functions → superposition principle for linerar systems*



Notice the Gibbs phenomenon at the jump discontinuity of the periodic function

- ***Non-periodic forcing***:
  - *Anlaytical methods: covolution integrals, Laplace transform, … → may become tedious or even impossible to solve depending on the forcing*
  - *Numerical methods: Runge-Kutta methods, …*

Multi Degree of Freedom Systems

Multi-degree of freedom systems: MDOF

Most practical engineering systems are continuous and have an infinite number of degree of freedom.

However:

The vibration analysis of continuous systems requires the solution of partial differential equations → difficult and/or time consuming.

A solution:

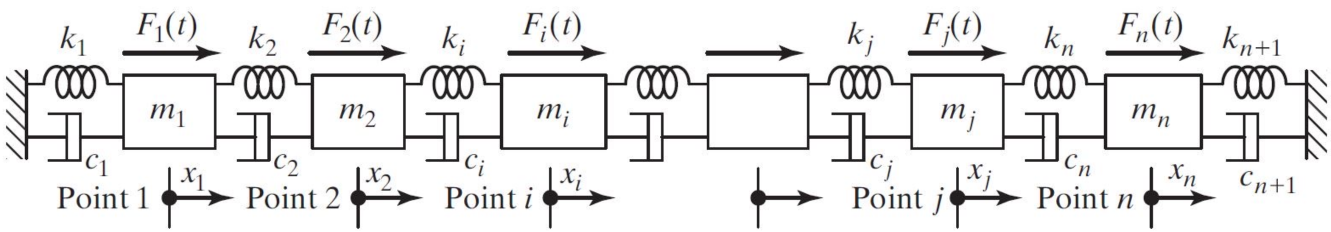
The analysis of multi-degree of freedom system only requires the solution of a set of ordinary differential equations → easier

So:

Continuous systems are often approximated as multi-degree of freedom systems.

Multi Degree of Freedom Systems

Multi-degree of freedom systems: MDOF



- *The equation of motion reads for the mass m_i reads:*

$$m_i \ddot{x}_i = \sum_j F_{ij}$$

$$m_i \ddot{x}_i = -k_i(x_i - x_{i-1}) + k_{i+1}\,(x_{i+1} - x_i) - c_i(\dot{x}_i - \dot{x}_{i-1}) + c_{i+1}\,(\dot{x}_{i+1} - \dot{x}_i) + F_i$$

- *Expressing the equation of motion of the complete system in matrix form yields a system of ordinary differential equations*

$$[M]\ddot{x} + [C]\dot{x} + [K]x = F$$

$[M]$, $[C]$ *and* $[K]$ *are the mass, damping and stiffness matrices, respectively. They are symmetric.*

Multi Degree of Freedom Systems

Multi-degree of freedom systems: MDOF

- *Mass, damping and stiffness matrices:*
  - *If the **mass matrix** is not diagonal → the system is said to have **mass (or inertia) coupling***
  - *If the **damping matrix** is not diagonal → the system is said to have **velocity coupling***
  - *If the **stiffness matrix** is not diagonal → the system is said to have **static coupling***

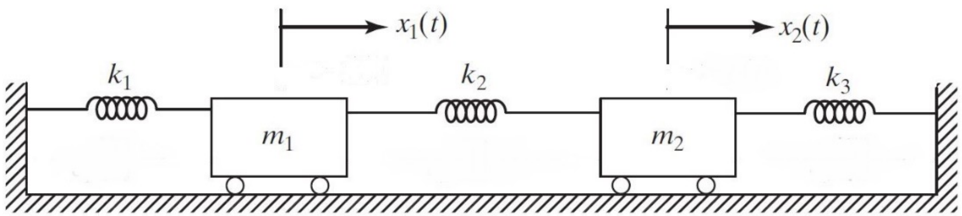
→ *If the system of equation is coupled, then the equations have to be solved simultaneously.*

$$[M]\ddot{x} + [C]\dot{x} + [K]x = F$$

Two Degree of Freedom Systems

2 DOF system: undamped free-vibration

- *Consider the following system:*



- *The equation of motion reads:*

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

- *As for the SDOF case, we can assume the masses undergo harmonic motion. The solution therefore takes the form: $x_j(t) = X_j e^{i\omega t}$. Substituting in the equation of motion:*

$$\begin{bmatrix} -\omega^2 m_1 + k_1 + k_2 & -k_2 \\ -k_2 & -\omega^2 m_2 + k_2 + k_3 \end{bmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Two Degree of Freedom Systems

2 DOF system: undamped free-vibration

- *For the system to have a non-trivial solution, the determinent of the coeffcients of X_1 and X_2 must be zero:*

$$det\begin{bmatrix} -\omega^2 m_1 + k_1 + k_2 & -k_2 \\ -k_2 & -\omega^2 m_2 + k_2 + k_3 \end{bmatrix} = 0$$

Or

$$(m_1 m_2)\omega^4 - \big((k_1 + k_2)m_2 + (k_2 + k_3)m_1\big)\omega^2 + \Big((k_1 + k_2)(k_2 + k_3) - k_2^2\Big) = 0$$
$$\Leftrightarrow A\omega^4 + B\omega^2 + C = 0$$

- *The solution to the above equation yields the natural frequncies of the system: ω_1^2 and ω_2^2.*

$$\omega_{1,2}^2 = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

- *The values of X_1 and X_2 remain to determined, but it is out of the scope of this course.*

Two Degree of Freedom Systems

2 DOF system: the general case → damped with harmonic forcing

- *The genral equation of motion reads:*

$$\begin{bmatrix} m_{11} & m_{12} \\ m_{12} & m_{22} \end{bmatrix} \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} + \begin{bmatrix} c_{11} & c_{12} \\ c_{12} & c_{22} \end{bmatrix} \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} + \begin{bmatrix} k_{11} & k_{12} \\ k_{12} & k_{22} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}$$

- *The forcing is assumed harmonic: $F_j = F_{0j} e^{i\omega t}$. The steady-state solution therefore takes the form: $x_j(t) = X_j e^{i\omega t}$. Substituting in the equation of motion:*

$$\begin{bmatrix} (-\omega^2 m_{11} + i\omega c_{11} + k_{11}) & (-\omega^2 m_{12} + i\omega c_{12} + k_{12}) \\ (-\omega^2 m_{12} + i\omega c_{12} + k_{12}) & (-\omega^2 m_{22} + i\omega c_{22} + k_{22}) \end{bmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} F_{01} \\ F_{02} \end{pmatrix}$$

$$\begin{bmatrix} Z_{11}(i\omega) & Z_{12}(i\omega) \\ Z_{12}(i\omega) & Z_{22}(i\omega)) \end{bmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} F_{01} \\ F_{02} \end{pmatrix}$$

- *$Z_{mn}(i\omega)$ is the mechanical impedance.*

Two Degree of Freedom Systems

2 DOF system: the general case → damped with harmonic forcing

- *The equation can be solved as:*

$$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{bmatrix} Z_{11}(i\omega) & Z_{12}(i\omega) \\ Z_{12}(i\omega) & Z_{22}(i\omega)) \end{bmatrix}^{-1} \begin{pmatrix} F_{01} \\ F_{02} \end{pmatrix}$$

Which leads to:

$$X_1(i\omega) = \frac{Z_{22}(i\omega)F_{01} - Z_{12}(i\omega)F_{02}}{Z_{11}(i\omega)Z_{22}(i\omega) - Z_{12}^2(i\omega)}$$

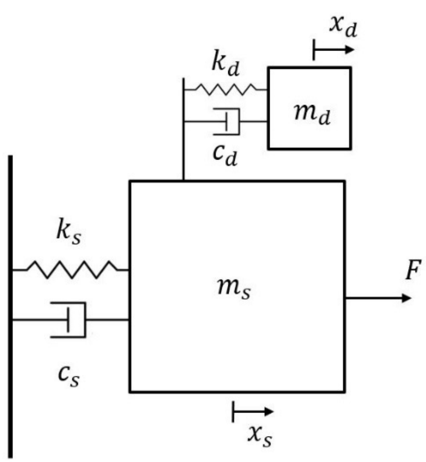
$$X_2(i\omega) = \frac{Z_{11}(i\omega)F_{02} - Z_{12}(i\omega)F_{01}}{Z_{11}(i\omega)Z_{22}(i\omega) - Z_{12}^2(i\omega)}$$

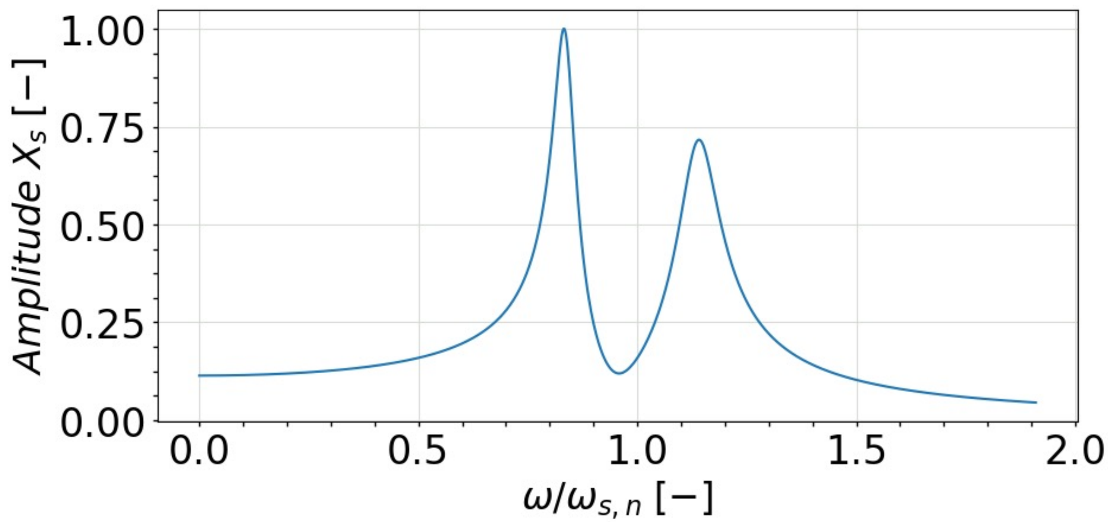
- *The complete steady state solution can be found by injecting X_1 and X_2 into $x_j(t) = X_j e^{i\omega t}$*
- *The frequency response curves can be found by considering the moduli of $X_1(i\omega)$ and $X_2(i\omega)$.*

Two Degree of Freedom Systems

2 DOF system: the general case → damped with harmonic forcing

- *Frequency response curves of a 2DOF system:* $F = F_0 e^{i\omega t}$





Numerical Approach

MDOF systems: let's symplify and consider a 2 DOF system

An alternative to analytically solving the systems: numerically solving the systems

→ *The state space representation:*

- *Converts the system into a 1st order ODE system*
- *Formulation that allows for an easier numerical solution*

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \ddot{x}_1\\ \ddot{x}_2\end{bmatrix} = \begin{bmatrix}\begin{matrix}0 & 0\\ 0 & 0\end{matrix} & \begin{matrix}1 & 0\\ 0 & 1\end{matrix}\\ [-\mathbf{M}^{-1}\boldsymbol{K}] & [-\mathbf{M}^{-1}\mathbf{C}]\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ \dot{x}_1\\ \dot{x}_2\end{bmatrix} + \begin{bmatrix}\begin{matrix}0 & 0\\ 0 & 0\end{matrix}\\ [\boldsymbol{M}^{-1}]\end{bmatrix}\begin{bmatrix}F_1\\ F_2\end{bmatrix}$$