Problem 1.

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When there are multiple choices in the following, select all statements that are true.

1. Consider the discrete-time switched system $x(k+1) = A_{\sigma(k)}x(k)$, $x(k) \in \mathbb{R}^3$ where $\sigma(k) \in \{1, 2\}$.

   ◯ If $\exists P \in \mathbb{R}^{3\times 3}$, $P = P^T > 0$ such that
   $$A_1^T P A_1 - P < 0 \quad \text{and} \quad A_2^T P A_2 - P < 0$$
   then the system is exponentially stable.

   ◯ If the system is exponentially stable, then there is $P \in \mathbb{R}^{3\times 3}$, $P = P^T > 0$ such that
   $$A_1^T P A_1 - P < 0 \quad \text{and} \quad A_2^T P A_2 - P < 0$$

   ◯ If $\mathrm{Spec}(A_1) = \{1 + j, 1 - j, 0\}$, then the system cannot be exponentially stable.

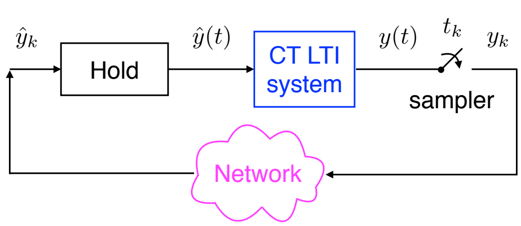
2. Consider the LTI system $x(k+1) = Ax(k)$, $x(k) \in \mathbb{R}^n$.

   ◯ If there is a matrix $P = P^T > 0$ such that $A^T P A - P = -I$, then, for all $x(0) \in \mathbb{R}^n$, $x(k)$ is bounded.

   ◯ If, for all $x(0) \in \mathbb{R}^n$, $\lim_{k\to+\infty} x(k) = 0$, then $\exists \alpha > 0, \rho \in [0, 1)$ such that $\|x(k)\| \le \alpha\rho^k \|x(0)\|$, $\forall x(0) \in \mathbb{R}^n$.

   ◯ If $x(k)$ is bounded for all $x(0) \in \mathbb{R}^n$, then there is a matrix $P = P^T > 0$ such that $A^T P A - P = -I$.

3. Consider the NCS given in the figure, characterized by packet dropouts, network-induced delay $\tau = 0$, and constant sampling period $T > 0$.



CT LTI system: $\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases}$

As seen in the lectures, the discrete-time NCS model is

$$z_{k+1} = \psi_{\theta_k} z_k, \qquad z_k = \begin{bmatrix} x_k \\ \hat{y}_{k-1} \end{bmatrix}, \qquad \psi_\theta = \begin{bmatrix} e^{AT} + \theta\Gamma(T-\tau)BC & e^{A(T-\tau)}\Gamma(\tau)B + (1-\theta)\Gamma(T-\tau)B \\ \theta C & (1-\theta)I \end{bmatrix}$$

where $\Gamma(s) = \int_0^s e^{At}dt$ and $\theta_k \in \{0, 1\}$, $k = 0, 1, 2, \ldots$. Assume that the asymptotic packet dropout rate $r \in [0, 1]$ exists.

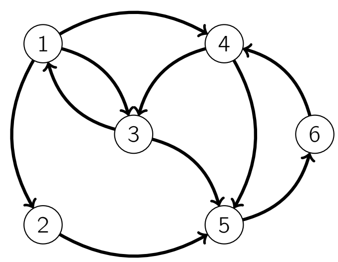
   ◯ If $r = 0.5$ and there are $P = P^T > 0$ and $\alpha, \alpha_0, \alpha_1$ such that
   $$\sqrt{\alpha_o \alpha_1} > \alpha > 1, \quad \psi_0^T P \psi_0 \le \alpha_0^{-2} P, \quad \psi_1^T P \psi_1 \le \alpha_1^{-2} P$$
   then the NCS is exponentially stable.

- ◯ If $r = 1$ the NCS cannot be asymptotically stable.
- ◯ If there is $P = P^T > 0$ such that

$$\psi_0^T P \psi_0 \leq e^3 P, \quad \psi_1^T P \psi_1 \leq e^{-\frac{1}{2}} P$$

then the NCS is exponentially stable for all $r < \dfrac{1}{7}$.

4. Let A be the binary adjacency matrix of the digraph in the figure.



- ◯ A is reducible.
- ◯ The element $(4, 2)$ of A^5 is nonzero.
- ◯ The element $(4, 2)$ of $\sum_{k=0}^{4} A^k$ is nonzero.

5. Let G be a weighted graph with $n > 2$ nodes and let L be its Laplacian matrix.

- ◯ If G is directed and a node v is a sink, then $L_{v,v} = 0$.
- ◯ If G is undirected and connected, then the eigenvalue $\lambda = 0$ of L has algebraic multiplicity 2.
- ◯ If G is directed and contains a globally reachable node, then all and only equilibria of $\dot{x} = -Lx$ are the states $\bar{x} = \alpha \mathbb{1}_n$, $\alpha \in \mathbb{R}$.

6. Let $T = (V, E)$ be an undirected graph with $V = \{1, 2, \ldots, 10\}$. Assume T is a tree.

- ◯ T has 10 edges.
- ◯ Removing an edge from T makes T disconnected.
- ◯ T is connected.

7. Assume that the digraph G with vertex set $V = \{1, 2, \ldots, 5\}$ is weakly connected and has a strongly connected component induced by the vertices $\{1, 2, 3\}$. Then,

- ◯ G contains the cycle $(1, 2)$, $(2, 3)$, $(3, 1)$.
- ◯ The subgraph induced by the set of vertices $\{1, 2, 4\}$ is strongly connected.
- ◯ The condensation graph of G has at least two nodes.

Problem 2.

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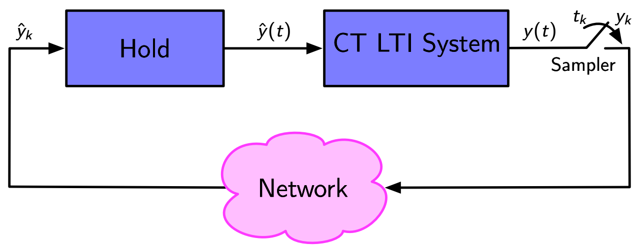
Consider the NCS in the figure below



where the LTI system is the first-order model

$$\begin{cases} \dot{x} = x + 2u \\ y = cx \end{cases}.$$

1. Assume that the sampling time $T = \log(4)$ is constant and the network induced delay $\tau < T$ is constant but unknown. Compute the inequalities characterizing all values of c and τ for which the NCS is asymptotically stable. Find then all values of c, if any, guaranteeing asymptotic stability for $\tau = \log(2)$.

   Hint: Recall the Jury's criterion: the roots of $\phi(\lambda) = \lambda^2 + \alpha\lambda + \beta$ verify $|\lambda| < 1$ if and only if

$$\begin{aligned} &\beta > -\alpha - 1 \\ &\beta > \alpha - 1 \\ &\beta < 1 \end{aligned}.$$

2. Assume now that $T = \log(4)$, $c = 1$ but the delay τ_k is time-varying in the interval $[0.1, 0.2]$. Give sufficient conditions for certifying the exponential stability of the NCS through a quadratic Lyapunov function.

Problem 3.

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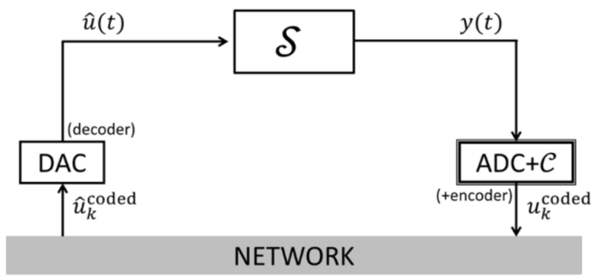
Consider the first-order continuous-time system $\mathcal{S}$ with dynamics

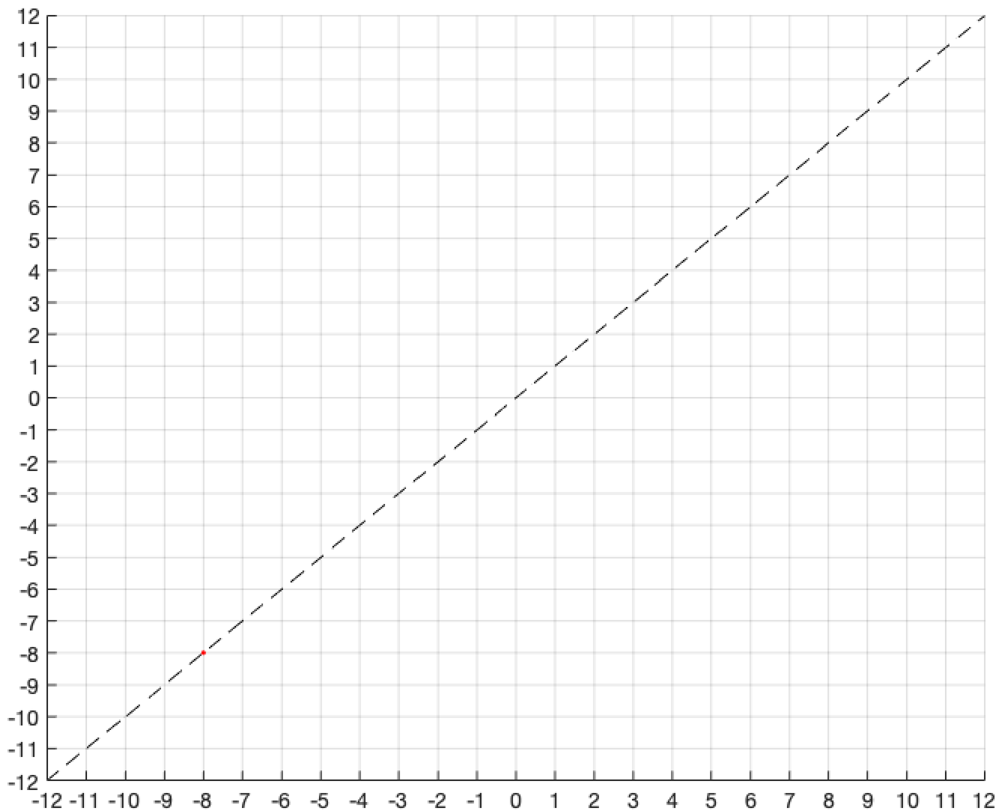
$$\dot{x} = \frac{1}{2}x + 2u$$

along with the NCS shown in the figure below, where the network is ideal. Assume the system is controlled in a sample-and-hold fashion, that is $u(t) = u_k$ for $t \in [kT, (k+1)T)$, $k \geq 0$. Moreover, assume that $u_k \in \mathcal{U}$, where $\mathcal{U}$ contains 2^{N_B} values.



1. Compute the minimun rate $\frac{N_B}{T}$ required to make the system boundable. For $T = 2\log(4) = 2.7726$ s, compute the minimum number of bits N_B to be transmitted within each sampling interval to make the system boundable.

2. Set $T = 2\log(4)$ s, and assume that $N_B = 1$. Using the control law $u_k = -\text{sign}(x_k)$, show the properties of the state trajectories using the graphical method. Plot the results in the graduated figure below, starting at least from $x_0 = 1$ and $x_0 = 2$.



3. Set $T = 2\log(4)$ s, $N_B = 2$ and $\mathcal{U} = \{-1, 1\}$. Compute the control law for guaranteeing boundability when one operates at the rate limits. Compute the corresponding positively invariant set.

Problem 4.

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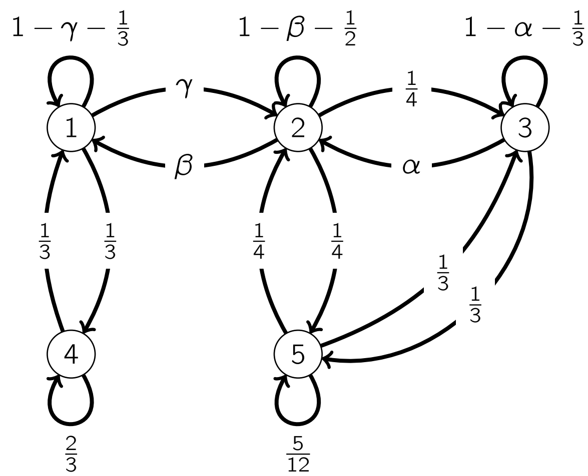
Let A be the adjacency matrix of the digraph G in the figure below and consider the discrete-time system $x(k+1) = Ax(k)$.



1. Set $\alpha = \beta = 0$. Compute all values of $\gamma > 0$ such that A is stochastic and the system reaches a consensus state, as $k \to +\infty$. Compute also the consensus value.

2. Compute the values of α, β, and γ such that the digraph G is the output of the Metropolis-Hasting algorithm.

3. Assume that nodes are sensors, each storing a noisy sample $y_i = \theta + v_i$ $i = 1, 2, \ldots, 5$ of a common scalar parameter θ. The random variables v_i are independent and Gaussian with zero mean and variance $\sigma_i^2 > 0$. Describe how to use the digraph G obtained in point 2 for computing the BLUE estimate $\hat{\theta}$ of θ through distributed computations. Provide also the expression of $\hat{\theta}$.

Problem 5.

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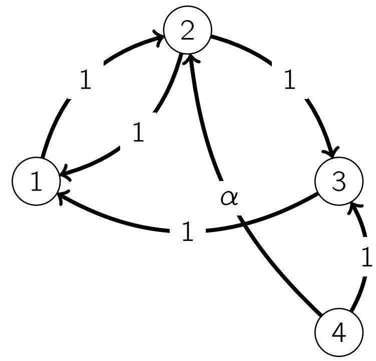
Consider the Laplacian flow $\dot{x} = -Lx$, $x(0)^T = \begin{bmatrix} 2 & 4 & 4 & 4 \end{bmatrix}$ where L is the Laplacian matrix of the following digraph and $\alpha > 0$.



1. Does $x(t)$ reach consensus as $t \to +\infty$? If yes, compute the consensus value.

2. Redefine the weights of the subgraph $\tilde{G}$ spanned by $\{1, 2, 3\}$ in order to obtain average consensus on $\tilde{G}$. Draw $\tilde{G}$.

Problem 6.

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Let G be an unweighted strongly connected digraph with binary adjacency matrix $A \in \{0,1\}^{n\times n}$, $n \geq 2$. Let (λ, v) be the dominant eigenpair, that is $Av = \lambda v$ and $\mathbb{1}_n^T v = 1$. Consider the matrix $P \in \mathbb{R}^{n\times n}$ with entries

$$P_{ij} = \frac{1}{\lambda}\frac{v_j}{v_i}A_{ij}, \quad i,j \in \{1,2,\dots,n\}$$

1. Show that $v \succ 0$ and that P is row stochastic and irreducible.

2. Pick $i,j \in \{1,2,\dots,n\}$ and $k \geq 1$. By considering the weighted digraph associated to P, assume there is a path of length k from i to j. Show that the product of the edge weights along the path is $\frac{1}{\lambda^k}\frac{v_j}{v_i}$.