Lecture 9

The Laplacian matrix and consensus in continuous time

Textbook , §6, §7

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Setup

- $G = (V, E, w)$ is a weighted digraph, $V = \{1, \dots, n\}$
  - A is the adjacency matrix
  - $D_{\text{out}} = \begin{bmatrix} d^{\text{out}}(1) & & 0 \\ & \ddots & \\ 0 & & d^{\text{out}}(n) \end{bmatrix}$ is the out-degree matrix
  - Standing assumption: $w_{ij} > 0$ if $(i, j) \in E$

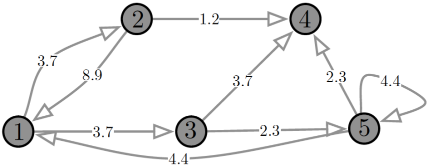
Definition

The *Laplacian* matrix of G is $L = D_{\text{out}} - A$

Remark

$$L_{ij} = \begin{cases} -A_{ij} & i \neq j \\ \displaystyle\sum_{h=1, h \neq i}^{n} A_{ih} & i = j \end{cases}$$

Example



$$L = \begin{bmatrix} 7.4 & -3.7 & -3.7 & 0 & 0 \\ -8.9 & 10.1 & 0 & -1.2 & 0 \\ 0 & 0 & 6.0 & -3.7 & -2.3 \\ 0 & 0 & 0 & 0 & 0 \\ -4.4 & 0 & 0 & -2.3 & 6.7 \end{bmatrix}$$

Properties of L

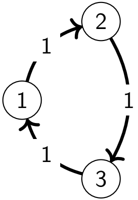
- Off-diagonal elements are ≤ 0. $L_{ii} \geq 0$
- $L_{ii} = 0$ only if i is a sink (up to a self-loop)
- Zero row-sums: $L\mathbb{1}_n = 0$
  - $\lambda = 0$ is an eigenvalue of L and $\mathbb{1}_n$ is the associated eigenvector
- Self-loops are invisible from L
- L symmetric $\Leftrightarrow$ A symmetric

Balanced graphs and undirected graphs

Definition: The digraph G is balanced if $d^{\text{out}}(v) = d^{\text{in}}(v), \ \forall v \in V$

G balanced $\not\Rightarrow$ A symmetric

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$



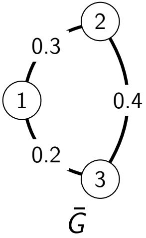
Definition: A digraph $G = (V, E, w)$ and an undirected, weighted graph $\bar{G} = (\bar{V}, \bar{E}, \bar{w})$ are associated with each other if $V = \bar{V}$ and

- $(i, j) \in \bar{E} \Rightarrow (i, j)$ and $(j, i) \in E$
- G has no self loops
- $\bar{w}_{ij} > 0 \Rightarrow w_{ij} = w_{ji} = \bar{w}_{ij}$

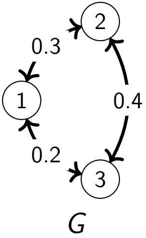
Proposition: Let G be associated to $\bar{G}$ and let A and $\bar{A}$ be the respective adjacency matrices. Then

- $A = \bar{A}$ and they are symmetric
- G is balanced

Balanced and undirected graphs



$\Rightarrow$



$\downarrow$ $\downarrow$

$$\bar{A} = \begin{bmatrix} 0 & 0.3 & 0.2 \\ 0.3 & 0 & 0.4 \\ 0.2 & 0.4 & 0 \end{bmatrix} \qquad A = \begin{bmatrix} 0 & 0.3 & 0.2 \\ 0.3 & 0 & 0.4 \\ 0.2 & 0.4 & 0 \end{bmatrix}$$

$\Rightarrow$ Same adjacency matrices $\rightarrow$ Laplacians are the same

Definition: The Laplacian matrix of an undirected weighted graph $\bar{G} = (V, \bar{E}, \bar{w})$ is the Laplacian matrix of the associated digraph.

Balanced and undirected graphs

Remark

$\bar{G}$ undirected and connected $\Rightarrow$ the associated digraph G is strongly connected (because of bidirectional edges)

Proposition

If the digraph G has a Globally Reachable Node (GRN) and is associated to an undirected graph $\bar{G}$, then

(a) G is strongly connected

(b) $\bar{G}$ is connected

Back to the general case

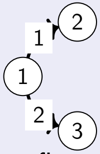
Definition: (Laplacian without using graphs) $L \in \mathbb{R}^{n \times n}$, $n \geq 2$ is a Laplacian matrix if

- all row sums are zero
- diagonal entries are ≥ 0
- off-diagonal entries are ≤ 0

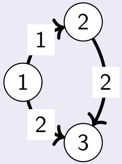
Remark: To every L one can associate a unique digraph, up to self-loops

Example:

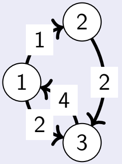
$$L = \begin{bmatrix} 3 & -1 & -2 \\ 0 & 2 & -2 \\ -4 & 0 & 4 \end{bmatrix}$$



From first row
(off-diagonal entries)



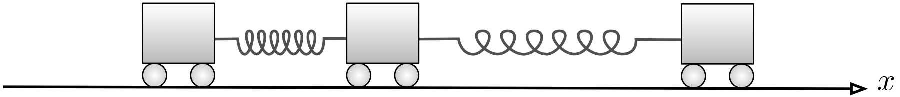
From second row



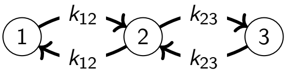
From third row

Examples of Laplacians in physical systems

String of masses connected by springs



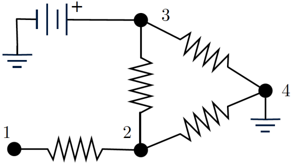
- Associated graph



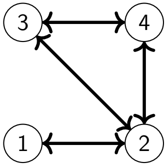
- Mass i subject to the elastic force: $F_i = \sum_{j \in \mathcal{N}^{out}(i)} k_{ij}(x_j - x_i) = -(Lx)_i$
- Total elastic energy: $E = \frac{1}{2} \sum_{(i,j) \in \mathcal{E}} k_{ij}(x_i - x_j)^2 = \frac{1}{2} x^T L x$
- Dynamics of the mass i: $M_i \ddot{x}_i = -(Lx)_i$, $M_i > 0$
  - Collective dynamics setting $x = [x_1, \dots, x_n]^T$, $M_i = 1$, $i = 1, \dots, n$

$$\ddot{x} = -Lx$$

Electrical network of resistors



- Associated coupling graph



- Current from i to j

$$C_{i\to j} = \frac{V_i - V_j}{r_{ij}} = a_{ij}(V_i - V_j)$$

- r_{ij} = resistance,

Weights a_{ij}

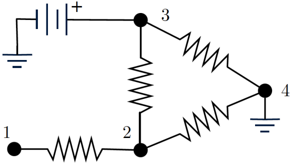
$\downarrow$

Adjacency matrix A

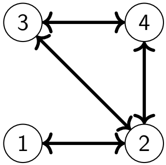
- Current injected into i: $c_i = -\sum_{j\in\mathcal{N}^{out}(i)} a_{ij}(V_i - V_j) = -(LV)_i$
    - Setting $V = [V_1, \dots, V_n]$ and $c = [c_1, \dots, c_n]$ one has

$$c = -LV$$

Electrical network of resistors



- Associated coupling graph



- Current from i to j

$$C_{i\to j} = \frac{V_i - V_j}{r_{ij}} = a_{ij}(V_i - V_j)$$

- r_{ij} = resistance,

Weights a_{ij}

$\downarrow$

Adjacency matrix A

- Power dissipated by a resistor: $C_{i\to j}(V_i - V_j)$
- Total dissipated power: $P = \sum_{(i,j)\in\mathcal{E}} a_{ij}(V_i - V_j)^2 = V^T L V$

Properties of Laplacian matrices

G : weighted digraph with n nodes $\rightarrow L$: Laplacian matrix, A : Adjacency matrix

Remark: G is balanced $\Leftrightarrow D_{out} = D_{in}$, where

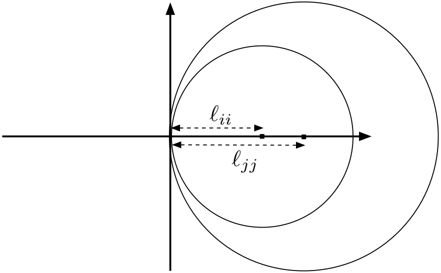
$$D_{\text{in}} = \begin{bmatrix} d^{\text{in}}(1) & & 0 \\ & \ddots & \\ 0 & & d^{\text{in}}(n) \end{bmatrix} \text{ is the in-degree matrix}$$

Lemma (zero column sum)

G is balanced $\Leftrightarrow \mathbb{1}_n^T L = [0, \ldots, 0]$

Proof: At home

Properties of Laplacian matrices



Lemma (spectrum of L)

The nonzero eigenvalues of L have strictly-positive real part

Proof: Recall the standing assumption that elements a_{ij} of the adjacency matrix are nonnegative. In row i of L, $l_{ii} = \sum_{j \neq i, j=1}^{n} a_{ij} \geq 0$, $l_{ij} = -a_{ij} \leq 0$

- Gersgorin Disks Theorem: $\mathrm{Spec}(L) \subset \cup_{i=1}^{n} D_i$
- $D_i = B(l_{ii}, r_i), r_i = \sum\limits_{i=1, j \neq i} |l_{ij}| = \sum_{i=1, j \neq i} a_{ij} = l_{ii}$

Properties of Laplacian matrices

Remark

- G undirected $\Rightarrow$ $L = L^T$ $\Rightarrow$ real eigenvalues. From the Lemma:

$$0 = \lambda_1 \leq \lambda_2 \cdots \leq \lambda_n$$

  - λ_2 is called the Fiedler eigenvalue
- If $\lambda = 0$ is simple, its eigenspace is the consensus subspace $\alpha \mathbb{1}_n, \alpha \in \mathbb{R}$. When $\lambda = 0$ is simple?

Theorem

Let d be the number of sinks in the condensation graph $C(G)$. Then, $\mathrm{rank}(L) = n - d$.

Remark

Recall the rank/nullity theorem: $\mathrm{rank}(L) + \dim(\mathrm{Ker}(L)) = n$, This shows that d is the geometric multiplicity of $\lambda = 0$.

Properties of Laplacian matrices

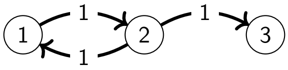
- **Theorem 1:** $d = 1 \;\Leftrightarrow\; G$ has a GRN (globally reachable node) $\Leftrightarrow$ the eigenvalue $\lambda = 0$ is simple
- **Corollary 2:** Assume G is undirected. Then $\lambda = 0$ is simple $\Leftrightarrow\; G$ is connected.

Proof of **Corollary 2**

Proof: Let $\tilde{G}$ be the digraph associated to G

- $\lambda = 0$ is simple $\Rightarrow d = 1$ (number of sinks of $C(\tilde{G})$) $\Rightarrow \tilde{G}$ is strongly connected $\Rightarrow G$ connected.
- G connected $\Rightarrow \tilde{G}$ strongly connected and $C(\tilde{G})$ has a single sink.

Example



- G has a globally reachable node $\Rightarrow \ \lambda = 0$ is a simple eigenvalue of L
- All other eigenvalues have positive real parts

Check:

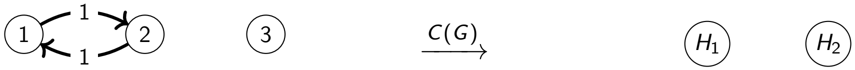
$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad L = D^{\text{out}} - A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{Spec}(L) = \{0, 0.382, 2.61\}$$

Computation of $Ker(L)$

$$L \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 0 \quad \begin{cases} v_1 - v_2 = 0 \\ 2v_2 - v_1 - v_3 = 0 \end{cases} \quad \rightarrow \quad \begin{cases} v_1 = v_2 \\ v_2 = v_3 \end{cases}$$

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} \alpha \\ \alpha \\ \alpha \end{bmatrix}, \quad \alpha \in \mathbb{R}$$

Example



- H_1 is the subgraph induced by nodes $\{1, 2\}$ and $H_2 = (\{3\}, \emptyset)$
- From the theorem, since $C(G)$ has two sinks
  - $\text{Rank}(L) = 3 - 2 = 1 \rightarrow \dim(\text{Ker}(L)) = 2$

Check:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad L = D^{\text{out}} - A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\text{Spec}(L) = \{2, 0, 0\}$$

$$L \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 0 \quad \begin{cases} v_1 - v_2 = 0 \\ -v_1 + v_2 = 0 \end{cases} \quad \rightarrow \quad \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} \alpha \\ \alpha \\ \beta \end{bmatrix}, \quad \alpha, \beta \in \mathbb{R}$$

The Laplacian flow

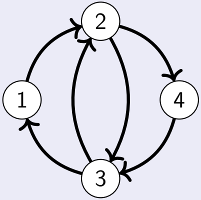
$$\dot{x} = -Lx \tag{1}$$

$x(t) \in \mathbb{R}^n,\ L \in \mathbb{R}^{n\times n}$ Laplacian matrix

Why is it interesting?

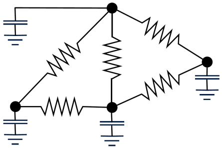
Agent dynamics: from (1), assuming unit weights,

$$\dot{x}_i = -\sum_{j\in\mathcal{N}^{\text{out}}(i)} (x_i - x_j)$$



- Agent i receives information only from its out neighbors
  - partial communication, distributed computations
- For $t \to +\infty$, do we have $x(t) \to$ *consensus state*?
  - if yes, when average consensus is achieved?
  - emergent behavior!

A physical example: RC network



- $c = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$ = injected current in the nodes
- $V = \begin{bmatrix} V_1 \\ \vdots \\ V_n \end{bmatrix}$ = voltages at the nodes

- We have seen that $c = -LV$, where L = Laplacian of the coupling graph with weights $a_{ij} = \frac{1}{r_{ij}}$
- If $C_1, \dots, C_n$ are the capacitances, then

$$C_i \dot{V}_i = c_i$$

The collective model, (for $C_i = 1,\ i = 1,..,n$)

$$\dot{V} = c \quad \rightarrow \quad \dot{V} = -LV$$

Problem:

- Will $x(t)$ converge to a consensus point $\alpha \mathbb{1}_n$ for some $\alpha \in \mathbb{R}$?
- Will $x(t)$ converge to average consensus, i.e. $\alpha =< x(0) >$?

Analysis of $\dot{x} = -Lx$

Lemma (equilibria)

If G contains a GRN, then all and only equilibria of $\dot{x} = -Lx$ are the states $\bar{x} = \alpha \mathbb{1}_n, \alpha \in \mathbb{R}$

Proof: $0 = -L\bar{x} \Leftrightarrow \bar{x}$ is the eigenvector with $\lambda = 0$. But

- $\alpha \mathbb{1}_n$ are eigenvectors of $\lambda = 0$
- $\lambda = 0$ is simple if G has a GRN
  - $\alpha \mathbb{1}_n$ are the only eigenvectors for $\lambda = 0$

Problem:

Is the set of equilibria attractive?

- Analysis of $x(t) = e^{-Lt}x(0)$

Analysis of $\dot{x} = -Lx \rightarrow x(t) = e^{-Lt}x(0)$

Theorem (Consensus with a GRN)

If G has a GRN, then

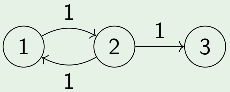
1. $\lim_{t\rightarrow+\infty} e^{-Lt} = \mathbb{1}_n w^T$, where w is the left eigenvector of L with $\lambda = 0$ verifying $w^T \mathbb{1}_n = 1$
2. $w \succcurlyeq 0$ and $w_i > 0$ if and only if the node i is globally reachable
3. the solution $\dot{x}(t) = -Lx(t)$ verifies
$$\lim_{t\rightarrow+\infty} x(t) = (w^T x(0))\mathbb{1}_n$$
4. if, in addition, G is balanced, then
   (a) G is strongly connected,
   (b) $\mathbb{1}_n^T L = [0 \dots 0]$,
   (c) $w = \frac{1}{n}\mathbb{1}_n$, and
   (d) $\lim_{t\rightarrow+\infty} x(t) = (\frac{1}{n}\mathbb{1}_n^T x(0))\mathbb{1}_n$

Analysis of $\dot{x} = -Lx$

Remarks:

- (3) is consensus
- (4-d) is average consensus
- G is strongly connected $\Leftrightarrow$ every node is globally reachable $\Leftrightarrow$ $w_i > 0 \ \ \forall \ i = 1, \ldots, n$

Example (consensus with a leader)



$$\dot{x}_i = - \sum_{j \in \mathcal{N}^{\text{out}}(i)} (x_i - x_j)$$

From the theorem

- $w^T = \begin{bmatrix} 0 & 0 & w_3 \end{bmatrix}$ $\quad w_3 > 0$ because only node 3 is GR
- $x(t) \to \left(w^T x(0)\right) \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = w_3 x_3(0) \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

  Consensus not influenced by agents 1 and 2

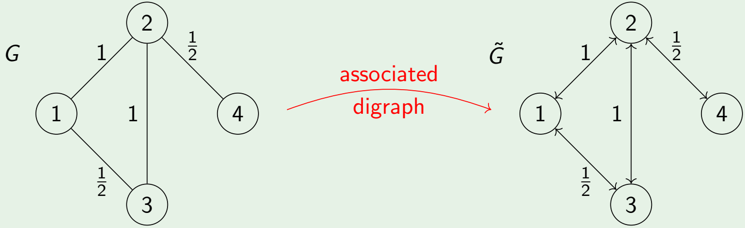
Check

$\dot{x} = -Lx$ gives

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = - \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \to \begin{cases} \dot{x}_1 = x_2 - x_1 \\ \dot{x}_2 = (x_3 - x_2) + (x_1 - x_2) \\ x_3(t) = x_3(0) \end{cases}$$

Example (average consensus)

Example (undirected graph)



G connected $\Rightarrow \tilde{G}$ is strongly connected and balanced

For $\dot{x} = -Lx$ one has $x(t) \to \left(\frac{1}{n}\mathbb{1}_n^T x(0)\right)\mathbb{1}_n$, i.e. average consensus

Design of balanced digraphs

Problem

Given a strongly connected and weighted digraph $G = (V, E, a)$, how to re-define the weights in order to obtain $\tilde{G} = (V, E, \tilde{a})$ that is balanced?

From the previous Theorem, there is a left eigenvector of L (the Laplacian associated to G) with zero eigenvalue verifying $w^T \mathbb{1}_n = 1$ and $w \succ 0$. So we have

$$L\mathbb{1}_n = 0 \text{ and } w^T L = 0^T$$

Define $L_{\text{res}} = \text{diag}(w)L$
We have $L_{\text{res}}\mathbb{1}_n = \text{diag}(w)L\mathbb{1}_n = 0$ and

$$\mathbb{1}_n^T L_{\text{res}} = \mathbb{1}_n^T \text{diag}(w)L = \begin{bmatrix} 1 & \dots & 1 \end{bmatrix} \begin{bmatrix} w_1 & & \\ & \ddots & \\ & & w_n \end{bmatrix} L = w^T L = 0$$

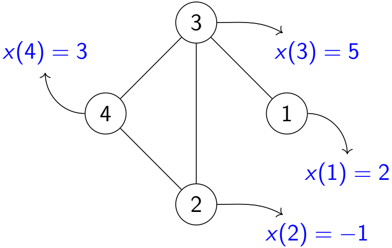
Remark

- L_{res} is a Laplacian matrix (zero row sum, positive diagonal entries, non-positive non-diagonal entries)
- L_{res} is the Laplacian of $\tilde{G}$ with weights $\tilde{a}_{ij} = w_i a_{ij}$
  $\hookrightarrow$ By construction $\tilde{G}$ is balanced!

Example

Laplacian flow for undirected, connected graphs - intuition

- $x = \left[\ x_1, \ldots, x_n\ \right]$ defines a function $x(v) \rightarrow \mathbb{R}$ on the node set V



Partial "derivative"

$$\partial_j x(i) = x(j) - x(i) \quad \text{if } j \in \mathcal{N}(i)$$

Properties

- $\partial_j x(j) = 0$
- $\partial_j x(i) = -\partial_i x(j)$
- $\partial_j^2 x(i) = \underbrace{\partial_j x(j)}_{=0} - \partial_j x(i) = x(i) - x(j)$

Laplacian operators

In calculus

$$\Delta f(\xi) \stackrel{\text{def}}{=} \sum_{i=1}^{n} \frac{\partial^2 f(\xi)}{\partial \xi_i^2} \quad \text{for } f : \mathbb{R}^n \to \mathbb{R}$$

On graphs

$$\Delta x(i) \stackrel{\text{def}}{=} -\sum_{j \in \mathcal{N}(i)} \partial_j^2(x(i)) = \sum_{j \in \mathcal{N}(i)} (x(j) - x(i))$$

L: Laplacian of the graph

Remark

$$\Delta x(i) = -(Lx)_i$$

Analogies

- In calculus $\Delta f(\xi) = 0$ if $f(\cdot)$ is constant
- On graphs $\Delta x(i) = 0$ if $x(\cdot)$ is constant (that is $x = \alpha \mathbb{1}_n$, $\alpha \in \mathbb{R}$)

- In calculus

$$\dot{f}(t,\xi) = \Delta f(t,\xi) \quad \text{THE HEAT EQUATION}$$

  - evolution of the temperature f at time t and in point ξ of an isolated room
    $\hookrightarrow f(t,\xi) \to \bar{f}(\xi)$ as $t \to +\infty$ where $\bar{f}$ constant in space (heat diffusion). Moreover $\bar{f}(\xi) = < f(0,\xi) >$, where $f(0,\xi)$ is the initial temperature in the point ξ

- On connected graphs

$$\dot{x}(t,v) = \Delta x(t,v) \leftrightarrow \dot{x}(t) = -Lx(t)$$

  gives $x(t) \to \bar{x}$ as $t \to +\infty$ where $\bar{x}$ in constant over the graph nodes ("space") and $\bar{x} =< x(0) >$

ALL ANALOGIES hold because Δ and $-L$ have a similar eigenstructure which qualifies $-L$ as a diffusion operator on graphs

Take home messages

- Graph Laplacians are the key for analyzing consensus algorithms in continuous time
  - continuous-time networks abstract real networks for very small sampling times
  - Laplacians naturally appear in physical models of electric and mechanical systems
- Consensus theorem for networks with a GRN
  - generalizations to time-varying graphs exist
- The Laplacian matrix is a diffusion operator on graphs