

Lecture 7

Consensus: convergence rate and digraphs that are not strongly connected

Textbook §3.3.3, §5.1, §5.2

Giancarlo Ferrari Trecate¹

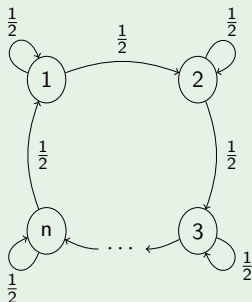
¹Dependable Control and Decision Group
École Polytechnique Fédérale de Lausanne (EPFL), Switzerland
`giancarlo.ferraritrecate@epfl.ch`

Analysis of $\rho_{\text{ess}}(A)$

- The smaller ρ_{ess} , the faster consensus
- $\rho_{\text{ess}}(A)$ depends upon the topology/weights of the digraph G associated to A
 - ↪ very difficult to compute, in general

Next: notable examples

Ring graph



to
from ↓

$$A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & \dots & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & \dots & 0 & 0 \\ \vdots & & & \ddots & \ddots & \\ \frac{1}{2} & 0 & 0 & \dots & 0 & \frac{1}{2} \end{bmatrix}$$

(A is a circulant matrix)

$$\text{Spec}(A) = \left\{ \frac{1}{2} + \frac{1}{2}e^{j\frac{2\pi h}{n}}, \quad h = 0, 1, \dots, n-1 \right\}$$

- G strongly connected, aperiodic $\rightarrow A$ primitive
- A doubly stochastic

$\hookrightarrow$ consensus theorem $\Rightarrow x(k) \rightarrow \langle x(0) \rangle \mathbb{1}_n$ as $k \rightarrow +\infty$

- $\lambda = 1$ is obtained for $h = 0$
- $\rho_{\text{ess}}(A)$ is obtained for $h = 1$

$$\rho_{\text{ess}}(A) = \left| \frac{1}{2} + \frac{1}{2}e^{j\frac{2\pi}{n}} \right| = \dots = \sqrt{\frac{1}{2} \left(1 + \cos \frac{2\pi}{n} \right)} \simeq$$

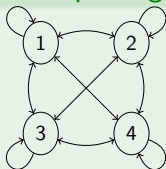
$$\cos(\theta) = 1 - \frac{\theta^2}{2} + O(\theta^4) \simeq \sqrt{1 - \frac{\pi^2}{n^2}} + O\left(\frac{1}{n^2}\right)$$

Taylor approx. for $\cos \theta$

Remarks

- $\rho_{\text{ess}} \rightarrow 1$ as n increases. Large $n \Rightarrow$ slow consensus
- The ring graph is one of the worst cases for consensus

Example: complete graph



weights: $\frac{1}{n}$

$$A = \frac{1}{n} \begin{bmatrix} 1 & \dots & 1 \\ \vdots & & \\ 1 & \dots & 1 \end{bmatrix}$$

- $\text{Spec}(A) = \{1, 0, \dots, 0\} \Rightarrow \rho_{\text{ess}}(A) = 0 \Rightarrow$ consensus in one step!
- Optimal for consensus but the worst for communication

Problem (weight assignment)

Given the digraph G , find weights ≥ 0 such that

- A is doubly stochastic and primitive
- A minimises $\rho_{\text{ess}}(A)$

↪ Very hard, in general. Requires numerical optimization

Main result from the previous lectures

Let $G = (V, E, w)$ be a digraph. Consider the consensus algorithm

$$x^+ = Ax \Rightarrow x(k) = A^k x(0) \quad A = \text{adjacency matrix of } G$$

Theorem (consensus with primitive, stochastic matrices)

If A is primitive and stochastic, the state trajectory $x(k)$ verifies

$$\lim_{k \rightarrow +\infty} x(k) = (w^T x(0)) \mathbb{1}_n \quad (1)$$

where w is defined in the usual way.

If, in addition, A is double stochastic, then $w = \frac{1}{n} \mathbb{1}_n$ and hence

$$\lim_{k \rightarrow +\infty} x(k) = \langle x(0) \rangle \mathbb{1}_n \quad (2)$$

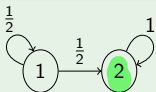
where $\langle x \rangle = \frac{1}{n} \sum_{i=1}^n x_i$

- A primitive $\Rightarrow G$ strongly connected
- Problem: what happens if G is not strongly connected?

Consensus in digraphs that are not Strongly Connected (SC)

$$x^+ = Ax$$

Example: opinion dynamics with a **single Globally Reachable (GR) node**



$$A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 1 \end{bmatrix} \succcurlyeq 0, \text{ stochastic, reducible } (n=2, r=2)$$

Powers of A

$$A^2 = \begin{bmatrix} \frac{1}{4} & 1 - \frac{1}{4} \\ 0 & 1 \end{bmatrix} \dots A^k = \begin{bmatrix} \frac{1}{2^k} & 1 - \frac{1}{2^k} \\ 0 & 1 \end{bmatrix}$$

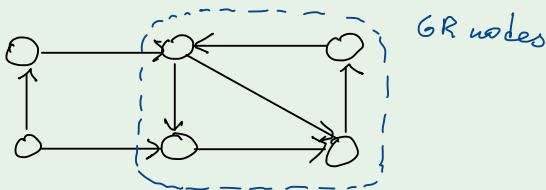
$$x(k) = A^k x(0) = A^k \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} \frac{1}{2^k} x_1(0) + \left(1 - \frac{1}{2^k}\right) x_2(0) \\ x_2(0) \end{bmatrix}$$

→ goes to $x_2(0)$
→ stubborn node not changing their opinion

Remarks

- There is a node not GR $\leftrightarrow$ A is reducible
- $A_{ij} > 0$ means agent j influences agent i (opposite direction of edges)
- 2 is GR. 2 acts as a leader. GR nodes = leaders? Yes, in general.

Example - multiple GR nodes



The "leader" is a subgraph. Consensus?

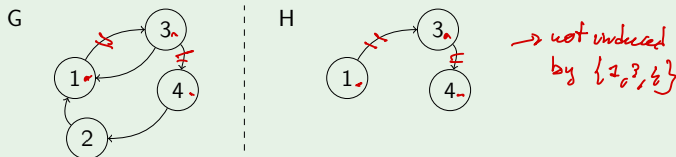
Subgraphs

Let $G = (V, E)$ be a digraph

Recall

$H = (V', E')$ is a subgraph of G if $V' \subseteq V$, $E' \subseteq E$ and E' connects only nodes in V'

Example

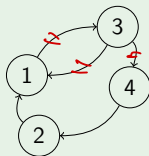


Definition

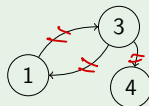
H is induced by $V' \subseteq V$ if E' contains all edges in E between vertices in V'

Example (ctd.) H induced by $V' = \{1, 3, 4\}$

G



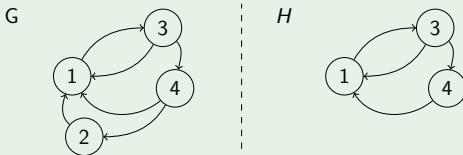
H



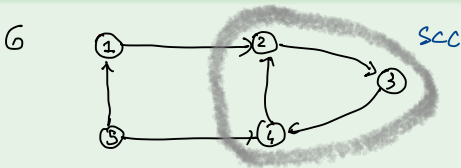
Definition

A subgraph H is a **strongly connected component (SCC)** of G if H is strongly connected (SC) and **any other subgraph of G strictly containing H is not SC**
with more nodes or edges

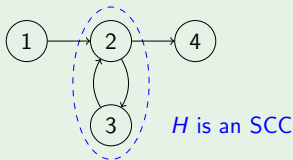
Example of a subgraph which is not an SCC



Example of an SCC

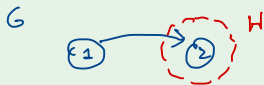


Example



SCC with more than one node: once a path enters the SCC, it can stay in there forever

Example: a sink is an SCC



• $H = (\{2\}, \emptyset)$ is SC

• any other node cannot be reached from 2 $\Rightarrow H$ cannot be contained in a larger SC subgraph of G

Lemma

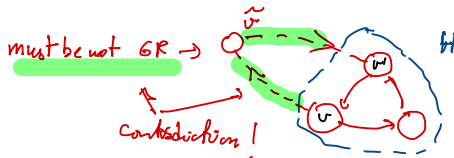
If a digraph G has a GR node, then the subgraph H induced by all GR nodes is an SCC of G .

shuvo $\downarrow$

Proof

- If H is made by a single node, it is SC and then an SCC
- If H is made by $k > 1$ nodes, let v and v' , $v \neq v'$ be two nodes. There is a path $v \dots v'$ and a path $v' \dots v$. Assume by contradiction that a path $v \dots v'$ crosses a node $\tilde{v} \notin H$. Then, $\tilde{v}$ is GR, which contradicts $\tilde{v} \notin H$.

We have shown that each path connecting nodes in H is contained in H . Therefore there is no digraph H' , SC and strictly including H



The condensation graph

Goal: given a digraph $G = (V, E)$ obtain a simplified digraph where each SCC has been collapsed into a node.

Definition

The condensation digraph $C(G)$ is defined as follows

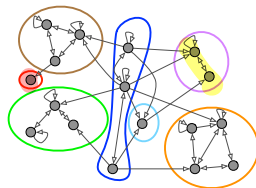
- Nodes $V(G)$ of $C(G)$ are the SCCs $H_1, \dots, H_s$ of G
- There is an edge (H_i, H_j) in $C(G)$ if there is an edge in G from H_i to H_j .

Example of condensation digraph

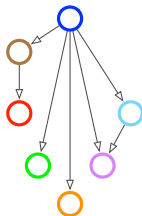
G



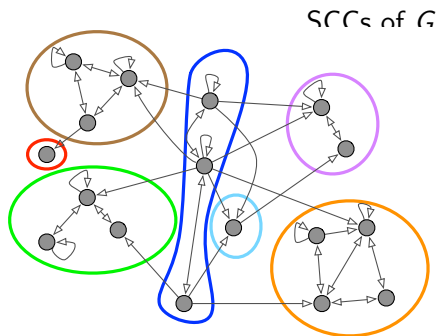
SCCs of G



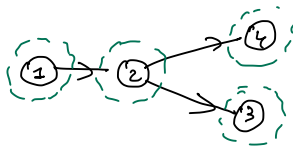
$C(G)$



A digraph can be always partitioned into SCCs, i.e. $\forall v \in V, \exists H \in V(G)$ s.t. $v \in H$



$\hookrightarrow$ In the "worst case" SCCs are composed by one node each



$\text{---} = \text{SCC}$

Lemma (properties of $C(G)$)

By construction,

- (i) $C(G)$ is **acyclic**
- (ii) $C(G)$ is **weakly connected only if G is weakly connected**
- (iii) The following statements are equivalent
 - (a) G has a GR node
 - (b) $C(G)$ has a GR node
 - (c) $C(G)$ has a **unique sink**

Remarks

Under (a), let H be the SCC of G induced by all GR nodes.

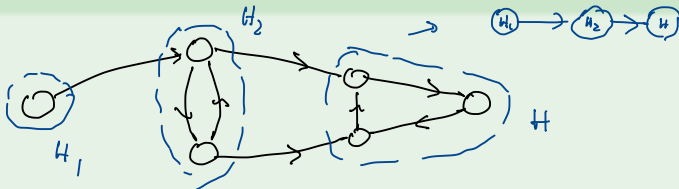
- Then H is the **unique sink of $C(G)$**
- Once in H , there is no way out

Lemma (properties of $C(G)$)

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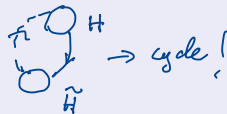
- i) $C(G)$ is acyclic
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Example

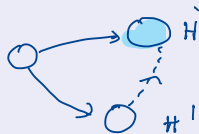


Proof (b) $\Rightarrow$ (c) (using the fact that $C(G)$ is acyclic)

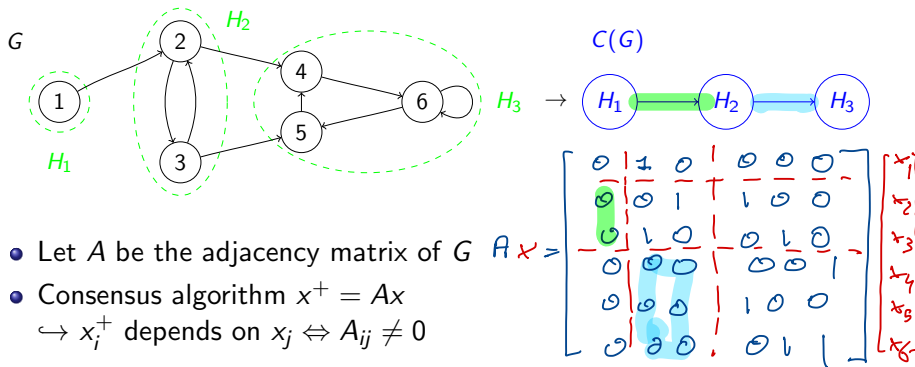
$\exists H \in C(G)$ which is GR. We show it is a sink. By contradiction, we assume H is not a sink. Then, there is an edge $(H, \tilde{H})$ and also a path from $\tilde{H}$ to H (as H is GR). But $C(G)$ has no cycles.



We now prove that the sink is unique. Assume, by contradiction, that there are 2 sinks H and H' . Since H is GR there is a path $H' \dots H$. But this cannot happen as H' has no outgoing edges.



Condensation digraphs and consensus



From $C(G)$ we deduce relations on blocks of states

$$\underbrace{\begin{bmatrix} x_4^+ \\ x_5^+ \\ x_6^+ \end{bmatrix}}_{H_3} \text{ does not depend on } \underbrace{\begin{bmatrix} x_2 \\ x_3 \end{bmatrix}}_{H_2} \text{ and } \underbrace{x_1}_{H_1}$$

$C(G)$



- $\begin{bmatrix} x_2^+ \\ x_3^+ \end{bmatrix}$ does not depend on x_1
- $\begin{bmatrix} x_2^+ \\ x_3^+ \end{bmatrix}$ depends on $\begin{bmatrix} x_4 \\ x_5 \\ x_6 \end{bmatrix}$
- ...

Conclusion: states in H_3 evolve autonomously and, over time, they will influence all other states.

- Let A_{H_3} be the adjacency matrix of H_3
 $\hookrightarrow$ If it is primitive and stochastic

$$\begin{bmatrix} x_4 \\ x_5 \\ x_6 \end{bmatrix} \rightarrow \alpha \mathbb{1}_3 \text{ as } k \rightarrow +\infty$$

for some α depending only on $x_4(0), x_5(0), x_6(0)$

- For $k \rightarrow +\infty$ nodes 2 and 3 receive α by the leaders $\{4, 5, 6\}$

One expects $\begin{bmatrix} x_2 \\ x_3 \end{bmatrix} \rightarrow \alpha \mathbb{1}_2$. True?

Similarly, node 1 receives α , as $k \rightarrow +\infty$. Is $x_1 \rightarrow \alpha$ as well?

and L10 ↓

Main result

Theorem (consensus with GR nodes)

Let $G = (V, E, w)$ be a digraph with row-stochastic adjacency matrix A . Assume G has a GR node and that the SCC H induced by all GR nodes is aperiodic. Then

- ① $\lambda = 1$ is a simple eigenvalue of A and it is strictly dominant, that is

$$\lambda = 1 > |\mu| \quad \forall \mu \in \text{Spec}(A), \mu \neq \lambda$$

- ② $\lim_{k \rightarrow +\infty} A^k = \mathbb{1}_n w^T$, where w is the left eigenvector of $\lambda = 1$ verifying $w^T \mathbb{1}_n = 1$

- ③ the elements of w verify

- $w_i > 0$ if $i \in V$ is GR
- $w_i = 0$ otherwise

Moreover, one has

$$\lim_{k \rightarrow +\infty} x(k) = (w^T x(0)) \mathbb{1}_n$$

Handwritten diagram illustrating the theorem conditions and the resulting consensus value:

A row vector $[x_1(0) \ x_2(0) \ x_3(0) \ x_4(0) \ x_5(0)]$ is shown. The first three elements are marked with red 'X' and labeled "not GR". The last two elements are marked with red circles and labeled "GR". A red arrow points from the "GR" label to the last two elements. To the right, a column vector $\begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \\ x_4(0) \\ x_5(0) \end{bmatrix}$ is shown, with the last two elements marked with red circles. The entire diagram is labeled with a red asterisk $(*)$.

$(**)$

Remarks

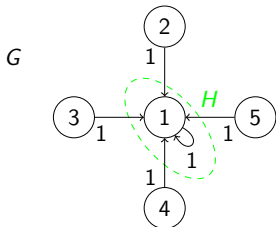
- The theorem provides a generalization of the consensus theorem for primitive and stochastic matrices.

$$A \text{ primitive} \Rightarrow G \text{ strongly connected} \Rightarrow H = G$$

- From (*) and (**) one has $w^T x(0)$ is a weighted average of the initial states of the leaders (leaders=GRNs)
- Generalisations to multiple subgraphs of "leaders"
↪ see the textbook

zhldo ↓

Example



① is GR. It shows in the picture

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \text{stochastic}$$

A stochastic and H aperiodic imply

$$\cdot w^T = [0 \ 0 \ 0 \ 0 \ 0] \stackrel{w^T \mathbb{1} = 1}{=} [1 \ 0 \ 0 \ 0 \ 0]$$

$$\cdot \text{from } (*), x(k) \rightarrow [x_1(0), x_1(0), x_1(0), x_1(0), x_1(0)]^T \text{ as } k \rightarrow \infty$$

Theorem (consensus with GR nodes)

Let $G = (V, E, w)$ be a digraph with row-stochastic adjacency matrix A . Assume G has a GR node and that the SCC H induced by all GR nodes is aperiodic. Then

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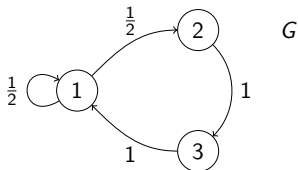
- ② $\lim_{k \rightarrow +\infty} A^k = \mathbb{1}_n w^T$, where w is the left eigenvector of $\lambda = 1$ verifying $w^T \mathbb{1}_n = 1$
- ③ the elements of w verify

- $w_i > 0$ if $i \in V$ is GR
 - $w_i = 0$ otherwise
- (*)

Moreover, one has

$$\lim_{k \rightarrow +\infty} x(k) = (w^T x(0)) \mathbb{1}_n \quad (**)$$

Example



• ④ is GR. Also ⑤, and ⑥

• $H = G \rightarrow \mathcal{C}(G) = (\{H\}, \emptyset)$

$$A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

• A is stochastic and H is aperiodic $\Rightarrow w^T = [* * *], * > 0$

$$\text{From } (**) \quad x(k) \rightarrow [* * *] \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix}$$

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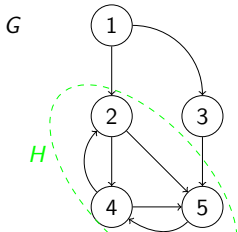
⑥ the elements of w verify

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- (*)

Moreover, one has

$$\lim_{k \rightarrow +\infty} x(k) = (w^T x(0)) \mathbb{1}_n \quad (**)$$

Example



⑤ is GR. H is the SCC

Assume A stochastic. Since H is aperiodic, then

$$w^T = [0 \quad * \quad 0 \quad * \quad *] \quad , \quad * > 0$$

$$A = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 1 \\ 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

From $(*)$, $x(k) \rightarrow (w^T x(0)) \mathbb{1}_5$

compute σ here w^T solving

using this adjacency matrix

$$\begin{cases} w^T A = w^T \\ w^T \mathbb{1}_5 = 1 \end{cases} \quad \text{linear system} \quad \text{in 3 unknowns}$$

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Moreover, one has

$$\lim_{k \rightarrow +\infty} x(k) = (w^T x(0)) \mathbb{1}_n \quad (**)$$

Summary on consensus (so far)

Properties of $x(\infty)$	Properties of A , stochastic	Properties of the digraph G	
Px1: Average consensus	← PA1: Doubly stochastic and primitive	→ PG1: SC (strongly connected) and aperiodic	Strong property
Px2: Consensus depending on all nodes ($w > 0$)	← PA2: Primitive		
Px3: Consensus that might depend only on some nodes		→ PG2: One aperiodic GR SCC	Weaker property

We have shown the following implications

- $PA1 \Rightarrow Px1$ and $PA1 \Rightarrow PG1$
- $PG1 \Rightarrow PA2$
- $PA2 \Rightarrow Px2$
- $PG2 \Rightarrow Px3$