

Lecture 7

Consensus: convergence rate and digraphs that are not strongly connected

Textbook §3.3.3, §5.1, §5.2

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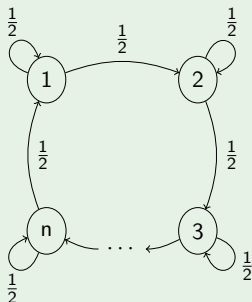
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Analysis of $\rho_{\text{ess}}(A)$

- The smaller ρ_{ess} , the faster consensus
- $\rho_{\text{ess}}(A)$ depends upon the topology/weights of the digraph G associated to A
 - $\hookrightarrow$ very difficult to compute, in general

Next: notable examples

Ring graph



$$A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & \dots & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & \dots & 0 & 0 \\ \vdots & & & \ddots & \ddots & \\ \frac{1}{2} & 0 & 0 & \dots & 0 & \frac{1}{2} \end{bmatrix}$$

(A is a circulant matrix)

$$\text{Spec}(A) = \left\{ \frac{1}{2} + \frac{1}{2} e^{j \frac{2\pi h}{n}}, \quad h = 0, 1, \dots, n-1 \right\}$$

- G strongly connected, aperiodic $\rightarrow A$ primitive
- A doubly stochastic

$\hookrightarrow$ consensus theorem $\Rightarrow x(k) \rightarrow \langle x(0) \rangle \mathbb{1}_n$ as $k \rightarrow +\infty$

- $\lambda = 1$ is obtained for $h = 0$
- $\rho_{\text{ess}}(A)$ is obtained for $h = 1$

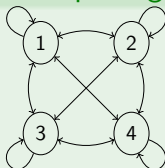
$$\rho_{\text{ess}}(A) = \left| \frac{1}{2} + \frac{1}{2} e^{j \frac{2\pi}{n}} \right| = \dots = \sqrt{\frac{1}{2} \left(1 + \cos \frac{2\pi}{n} \right)} \simeq$$

$$\underset{\text{Taylor approx. for } \cos \theta}{\simeq} \sqrt{1 - \frac{\pi^2}{n^2}} + O\left(\frac{1}{n^2}\right)$$

Remarks

- $\rho_{\text{ess}} \rightarrow 1$ as n increases. Large $n \Rightarrow$ slow consensus
- The ring graph is one of the worst cases for consensus

Example: complete graph



weights: $\frac{1}{n}$

$$A = \frac{1}{n} \begin{bmatrix} 1 & \dots & 1 \\ \vdots & & \\ 1 & \dots & 1 \end{bmatrix}$$

- $\text{Spec}(A) = \{1, 0, \dots, 0\} \Rightarrow \rho_{\text{ess}}(A) = 0 \Rightarrow$ consensus in one step!
- Optimal for consensus but the worst for communication

Problem (weight assignment)

Given the digraph G , find weights ≥ 0 such that

- A is doubly stochastic and primitive
- A minimises $\rho_{\text{ess}}(A)$

↪ Very hard, in general. Requires numerical optimization

Main result from the previous lectures

Let $G = (V, E, w)$ be a digraph. Consider the consensus algorithm

$$x^+ = Ax \Rightarrow x(k) = A^k x(0) \quad A = \text{adjacency matrix of } G$$

Theorem (consensus with primitive, stochastic matrices)

If A is primitive and stochastic, the state trajectory $x(k)$ verifies

$$\lim_{k \rightarrow +\infty} x(k) = (w^T x(0)) \mathbb{1}_n \quad (1)$$

where w is defined in the usual way.

If, in addition, A is double stochastic, then $w = \frac{1}{n} \mathbb{1}_n$ and hence

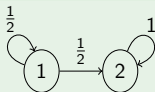
$$\lim_{k \rightarrow +\infty} x(k) = \langle x(0) \rangle \mathbb{1}_n \quad (2)$$

where $\langle x \rangle = \frac{1}{n} \sum_{i=1}^n x_i$

- A primitive $\Rightarrow G$ strongly connected
- **Problem: what happens if G is not strongly connected?**

Consensus in digraphs that are not Strongly Connected (SC)

Example: opinion dynamics with a single Globally Reachable (GR) node



Remarks

- There is a node not GR $\leftrightarrow A$ is reducible
- $A_{ij} > 0$ means agent j influences agent i (opposite direction of edges)
- 2 is GR. 2 acts as a leader. GR nodes = leaders? Yes, in general.

Example - multiple GR nodes

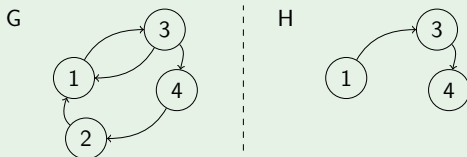
Subgraphs

Let $G = (V, E)$ be a digraph

Recall

$H = (V', E')$ is a subgraph of G if $V' \subseteq V$, $E' \subseteq E$ and E' connects only nodes in V'

Example

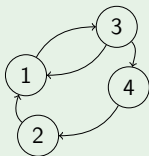


Definition

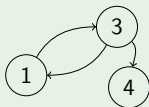
H is induced by $V' \subseteq V$ if E' contains all edges in E between vertices in V'

Example (ctd.) H induced by $V' = \{1, 3, 4\}$

G



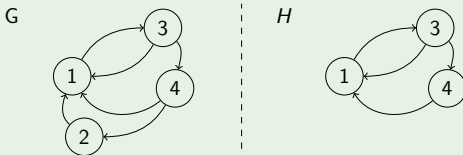
H



Definition

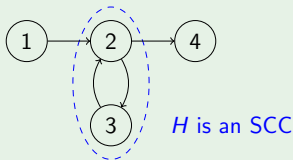
A subgraph H is a strongly connected component (SCC) of G if H is strongly connected (SC) and any other subgraph of G strictly containing H is not SC
with more nodes or edges

Example of a subgraph which is not an SCC



Example of an SCC

Example



SCC with more than one node: once a path enters the SCC, it can stay in there forever

Example: a sink is an SCC

Lemma

If a digraph G has a GR node, then the subgraph H induced by all GR nodes is an SCC of G .

Proof

- If H is made by a single node, it is SC and then an SCC
- If H is made by $k > 1$ nodes, let v and v' , $v \neq v'$ be two nodes.

There is a path $v \dots v'$ and a path $v' \dots v$.

Assume by contradiction that a path $v \dots v'$ crosses a node $\tilde{v} \notin H$.

Then, $\tilde{v}$ is GR, which contradicts $\tilde{v} \notin H$.

We have shown that each path connecting nodes in H is contained in H . Therefore there is no digraph H' , SC and strictly including H

The condensation graph

Goal: given a digraph $G = (V, E)$ obtain a simplified digraph where each SCC has been collapsed into a node.

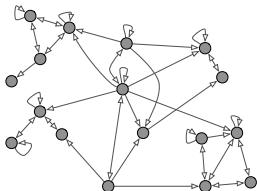
Definition

The condensation digraph $C(G)$ is defined as follows

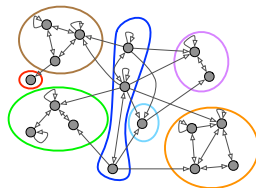
- Nodes $V(G)$ of $C(G)$ are the SCCs $H_1, \dots, H_s$ of G
- There is an edge (H_i, H_j) in $C(G)$ if there is an edge in G from H_i to H_j .

Example of condensation digraph

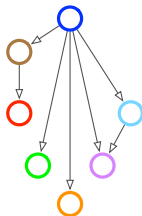
G



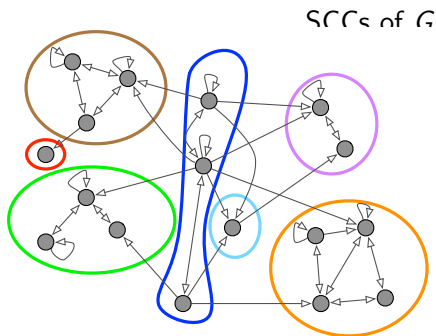
SCCs of G



$C(G)$



A digraph can be always partitioned into SCCs, i.e. $\forall v \in V, \exists H \in V(G)$
s.t. $v \in H$



$\hookrightarrow$ In the "worst case" SCCs are composed by one node each

Lemma (properties of $C(G)$)

By construction,

- (i) $C(G)$ is acyclic
- (ii) $C(G)$ is weakly connected only if G is weakly connected
- (iii) The following statements are equivalent
 - (a) G has a GR node
 - (b) $C(G)$ has a GR node
 - (c) $C(G)$ has a unique sink

Remarks

Under (a), let H be the SCC of G induced by all GR nodes.

- Then H is the unique sink of $C(G)$
- Once in H , there is no way out

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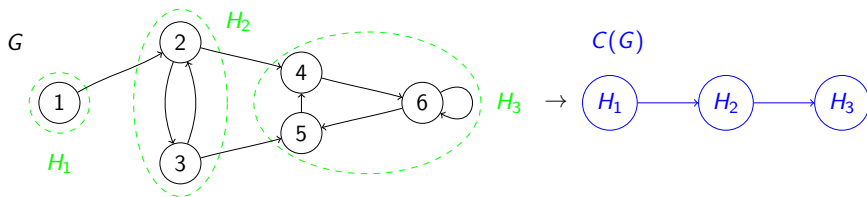
Example

Proof (b) $\Rightarrow$ (c) (using the fact that $C(G)$ is acyclic)

$\exists H \in C(G)$ which is GR. We show it is a sink. By contradiction, we assume H is not a sink. Then, there is an edge $(H, \tilde{H})$ and also a path from $\tilde{H}$ to H (as H is GR). But $C(G)$ has no cycles.

We now prove that the sink is unique. Assume, by contradiction, that there are 2 sinks H and H' . Since H is GR there is a path $H' \dots H$. But this cannot happen as H' has no outgoing edges.

Condensation digraphs and consensus



- Let A be the adjacency matrix of G
- Consensus algorithm $x^+ = Ax$
 $\hookrightarrow x_i^+$ depends on $x_j \Leftrightarrow A_{ij} \neq 0$

From $C(G)$ we deduce relations on blocks of states

$$\underbrace{\begin{bmatrix} x_4^+ \\ x_5^+ \\ x_6^+ \end{bmatrix}}_{H_3} \text{ does not depend on } \underbrace{\begin{bmatrix} x_2 \\ x_3 \end{bmatrix}}_{H_2} \text{ and } \underbrace{x_1}_{H_1}$$

$C(G)$



- $\begin{bmatrix} x_2^+ \\ x_3^+ \end{bmatrix}$ does not depend on x_1
- $\begin{bmatrix} x_2^+ \\ x_3^+ \end{bmatrix}$ depends on $\begin{bmatrix} x_4 \\ x_5 \\ x_6 \end{bmatrix}$
- ...

Conclusion: states in H_3 evolve autonomously and, over time, they will influence all other states.

- Let A_{H_3} be the adjacency matrix of H_3
 $\hookrightarrow$ If it is primitive and stochastic

$$\begin{bmatrix} x_4 \\ x_5 \\ x_6 \end{bmatrix} \rightarrow \alpha \mathbb{1}_3 \text{ as } k \rightarrow +\infty$$

for some α depending only on $x_4(0), x_5(0), x_6(0)$

- For $k \rightarrow +\infty$ nodes 2 and 3 receive α by the leaders $\{4, 5, 6\}$

One expects $\begin{bmatrix} x_2 \\ x_3 \end{bmatrix} \rightarrow \alpha \mathbb{1}_2$. **True?**

Similarly, node 1 receives α , as $k \rightarrow +\infty$. **Is $x_1 \rightarrow \alpha$ as well?**

Main result

Theorem (consensus with GR nodes)

Let $G = (V, E, w)$ be a digraph with row-stochastic adjacency matrix A . Assume G has a GR node and that the SCC H induced by all GR nodes is aperiodic.

Then

- ① $\lambda = 1$ is a simple eigenvalue of A and it is strictly dominant, that is

$$\lambda = 1 > |\mu| \quad \forall \mu \in \text{Spec}(A), \mu \neq \lambda$$

- ② $\lim_{k \rightarrow +\infty} A^k = \mathbb{1}_n w^T$, where w is the left eigenvector of $\lambda = 1$ verifying $w^T \mathbb{1}_n = 1$

- ③ the elements of w verify

- $w_i > 0$ if $i \in V$ is GR
 - $w_i = 0$ otherwise
- (*)

Moreover, one has

$$\lim_{k \rightarrow +\infty} x(k) = (w^T x(0)) \mathbb{1}_n \quad (**)$$

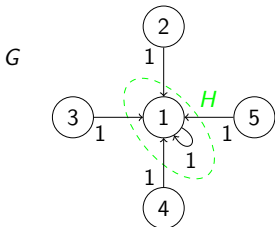
Remarks

- The theorem provides a generalization of the consensus theorem for primitive and stochastic matrices.

$$A \text{ primitive} \Rightarrow G \text{ strongly connected} \Rightarrow H = G$$

- From $(*)$ and $(**)$ one has $w^T x(0)$ is a weighted average of the initial states of the leaders (leaders=GRNs)
- Generalisations to multiple subgraphs of "leaders"
↪ see the textbook

Example



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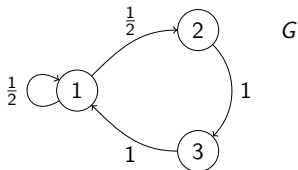
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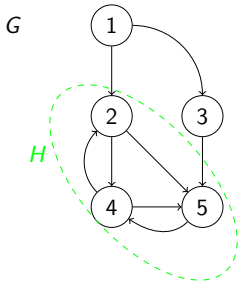
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Moreover, one has

$$\lim_{k \rightarrow +\infty} x(k) = \left(w^T x(0) \right) \mathbf{1}_n \quad (**)$$

Summary on consensus (so far)

Properties of $x(\infty)$	Properties of A , stochastic	Properties of the digraph G
Px1: Average consensus	PA1: Doubly stochastic and primitive	PG1: SC (strongly connected) and aperiodic
Px2: Consensus depending on all nodes ($w > 0$)	PA2: Primitive	
Px3: Consensus that might depend only on some nodes		PG2: One aperiodic GR SCC

We have shown the following implications

- $PA1 \Rightarrow Px1$ and $PA1 \Rightarrow PG1$
- $PG1 \Rightarrow PA2$
- $PA2 \Rightarrow Px2$
- $PG2 \Rightarrow Px3$