

Lecture 5

Networked control: coordination among agents Graph theory

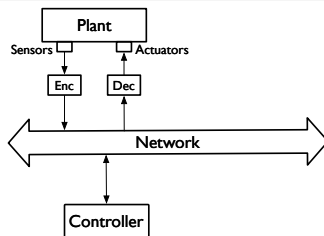
Giancarlo Ferrari Trecate¹

¹Dependable Control and Decision Group
École Polytechnique Fédérale de Lausanne (EPFL), Switzerland
giancarlo.ferraritrecate@epfl.ch

Timetable and Course Schedule (tentative)

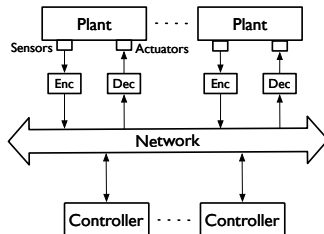
Part 1: Challenges (Week 1-6) - mostly 1 plant, 1 controller setting

- Review of LTI systems
- Review of Linear Matrix Inequalities (LMIs)
- Control networks and NCS
- Impact of delays
- Impact of packet drops



Part 2: Opportunities (Week 7-14) - multiple systems

- Coordination: motivating examples
- Elements of graph and matrix theory
- Discrete-time consensus
- Continuous-time consensus



Literature

- Opportunities in NCSs



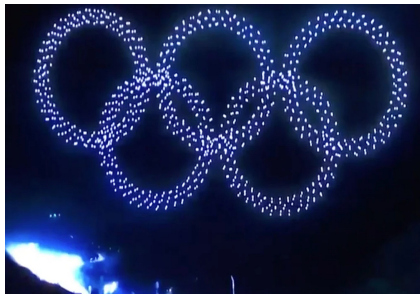
Francesco Bullo, Lecture notes on network systems, 2017. Available online at: <http://motion.me.ucsb.edu/book-1ns/>

From now on, this is called "THE textbook"

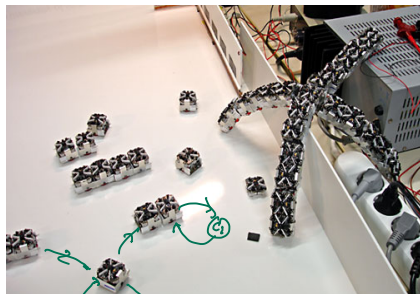


F. Garin and L. Schenato, "A Survey on Distributed Estimation and Control Applications Using Linear Consensus Algorithms," in Networked Control Systems, Springer London, pp.75-107, 2010.

Opportunities: coordination among agents



Drone show at the 2020 Olympic games



Swarm of mobile robots

Wishes

- Partial communication (limited transmission power)
- Distributed control
- Self-organizing for performing tasks

Coordination in nature

Social behavior: creatures cluster in large moving formations



School of fish



Swarm of flying birds

- Partial communication
- No centralized control
- **Global emergent behavior**

Outline

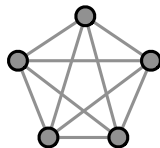
- Coordination: motivating problems in engineering and beyond (Textbook, 1.1-1.3)
- Modeling communication: graph theory (Textbook, Ch. 3)

Social influence networks - De Groot model

- n individuals, each with an estimate $p_i(0) \in \mathbb{R}$ of a common parameter
- individuals exchange information (communication network !)
- each individual i talks to all others and revises his estimate as

$$p_i^+ = \sum_{j=1}^n a_{ij} p_j$$

- $a_{ij} \geq 0$ are influence weights
- $\sum_{j=1}^n a_{ij} = 1$ (local averaging behavior)



Collective model

Set $p = [p_1 \ \dots \ p_n]^T$

$$p^+ = Ap, \quad A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$$

- DT LTI system
- A is *row-stochastic*, which means
 - ▶ non-negative entries
 - ▶ elements of each row sum up to 1

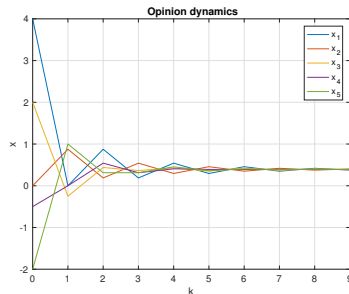
Problem

Will the team achieve *consensus*, i.e.

$$\exists \bar{p} \in \mathbb{R} : p_i(k) \rightarrow \bar{p} \text{ as } k \rightarrow +\infty, \forall i = 1, \dots, n ?$$

Simulations - De Groot model

$$p^+ = Ap, \quad A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1/4 & 0 & 1/4 & 1/4 & 1/4 \\ 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 1/3 & 1/3 & 0 & 1/3 \\ 0 & 1/2 & 1/2 & 0 & 0 \end{bmatrix}$$



Problems

- Properties of the weights a_{ij} for achieving consensus ?
- Can we predict the consensus value ?

Averaging in wireless sensor networks



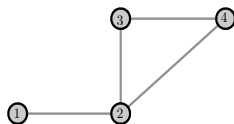
- n spatially distributed devices, each measuring the same environmental variable (temperature, light,...)
- devices exchange information over a communication network
- the operator wants to receive a single average measurement

Distributed algorithm

Sensor i computes

$$x_i^+ = \text{average}(x_i, x_j | j \sim i)$$

Example: $x_1^+ = \frac{x_1 + x_2}{2}$, $x_2^+ = \frac{x_1 + x_2 + x_3 + x_4}{4}$



$j \sim i \stackrel{\text{def}}{=} j \text{ is a neighbor of } i$
 $i \stackrel{\text{def}}{=} \text{the edge } (i, j) \text{ exists}$

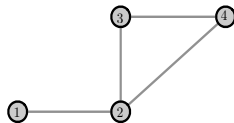
Collective model for the graph in the figure

$$\text{Set } x = [x_1 \quad \dots \quad x_n]^T$$

$$x^+ = Ax$$

$$A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

A is again row-stochastic



$j \sim i \stackrel{\text{def}}{=} j \text{ is a neighbor of } i$
 $i \stackrel{\text{def}}{=} \text{the edge } (i, j) \text{ exists}$

Problem

Will the sensors achieve **average consensus**, i.e.

$$x_i(k) \rightarrow \text{average}(x_i(0), i = 1, \dots, n) \text{ as } k \rightarrow +\infty, \forall i = 1, \dots, n ?$$

Remark: communication among sensors is just partial (e.g. 1 not connected to 4)

Alignment in teams of moving agents

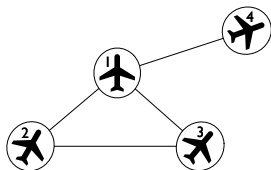
Set of n moving agents

- Dynamics of agent i :

$$v_i^+ = v_i + u_i \quad v_i(0) = \tilde{v}_i$$

- Velocity : $v_i(k) \in \mathbb{R}^2$

Control input : $u_i(k) \in \mathbb{R}^2$



Communication network topology:
undirected connected graph

$G = (V, E)$. Nodes $V = \{1, \dots, n\}$,
edges $E \subset V^2$.

Partial communication network used for computing the control law

$$u_i = \text{average}(v_i, v_j | j \sim i) - v_i$$

Collective model

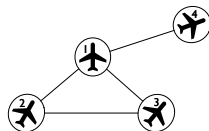
Set $v = [v_1^T \ \dots \ v_n^T]^T$

$$v^+ = Av$$

A composed by 2×2 blocks

$$A_{ij} = \begin{cases} \frac{1}{(\# \text{ neighbors to } i)+1} I_{2 \times 2} & \text{if } j \sim i \\ 0_{2 \times 2} & \text{otherwise} \end{cases}$$

A is row-stochastic



Problems

- Will the agents converge to a formation (all agent moving with the same velocity), i.e.

$$\exists \bar{v} \in \mathbb{R}^2 : v_i(k) \rightarrow \bar{v} \text{ as } k \rightarrow +\infty, \forall i = 1, \dots, n ?$$

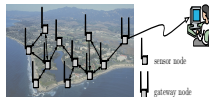
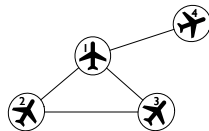
- Is $\bar{v} = \text{average}(v_i(0), \forall i = 1, \dots, n)$?

Common features to all examples

- The communication topology is captured by a graph G
- DT LTI collective dynamics

$$x^+ = Ax$$

where the structure of A depends on G



Problem

Is it possible to study consensus by analyzing the graph G ?

Next: elements of

- graph theory
- *algebraic* graph theory = how to relate graph and matrix properties

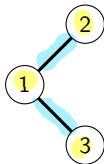
Graph theory

Basic definitions

Undirected graph

An *undirected* graph is a pair $G = (V, E)$ consisting of a finite set of *vertices* (or *nodes*), $V = \{1, 2, \dots, n\}$ and a set $E \subset V \times V$ of *unordered* pairs called *edges* (or *arcs*)

Example: $V = \{1, 2, 3\}$, $E = \{(1, 2), (1, 3)\}$



Undirected edges : $(1, 2) = (2, 1)$, $(1, 3) = (3, 1)$

Important convention

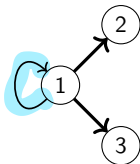
Self-loops (v, v) are NOT allowed in undirected graphs

Basic definitions

Directed graph

A graph G is *directed* (or *digraph*) if all pairs in E are *ordered*

Example: $V = \{1, 2, 3\}$, $E = \{(1, 1), (1, 2), (1, 3)\}$



Ordered edges: $(1, 2) \neq (2, 1)$, $(1, 3) \neq (3, 1)$

Remark

Self-loops (v, v) are allowed in directed graphs

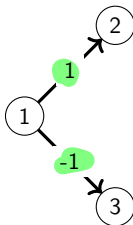
Basic definitions

Weighted graph

$$G = (V, E, w)$$

If a function $w : E \rightarrow \mathbb{R}$ is specified, the **graph/digraph** is called *weighted*

Example: $V = \{1, 2, 3\}$, $E = \{(1, 2), (1, 3)\}$ $w(1, 2) = 1$ and $w(1, 3) = -1$



Remarks

- Notation overload: $w_{ij} = w(i, j)$
- An **unweighted graph** is assimilated to a **weighted graph** with $w_{ij} = 1$, $\forall (i, j) \in E$

Undirected graphs

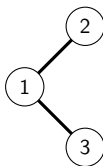
Undirected graphs: neighbors of a node

Let $G = (V, E)$ be an undirected graph

- Nodes u and v are neighbors if $(u, v) \in E$
 - Set of neighbors of u : $\mathcal{N}(u) = \{v \in V : (u, v) \in E\}$
 - Degree of u :

$$d(u) = \sum_{j \in \mathcal{N}(u)} w_{uj}$$

In unweighted graphs $d(u)$ counts the number of neighbors of u



$$u=1 \quad \mathcal{N}(1) = \{2, 3\} \quad d(1) = 2$$

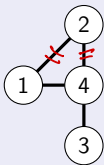
$$u=3 \quad \mathcal{N}(3) = \{1\} \quad d(3) = 1$$

Undirected graphs: subgraphs

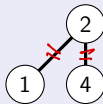
Subgraph

The graph $H = (U, F)$ is a *subgraph* of $G = (V, E)$ if $U \subseteq V$, $F \subseteq E$ and edges in F connect only nodes in U

Graph $G = (V, E)$



Graph $H = (U, F)$



$H = (U, F)$ is a subgraph of G because $U = \{1, 2, 4\} \subseteq V$,
 $F = \{(1, 2), (2, 4)\} \subseteq E$ and edges in F connects only nodes in U .

Spanning subgraph

A subgraph is *spanning* if its node set is V

th 67 d

Connectivity in *undirected* graphs

Path

A sequence of arcs $e_1 e_2 \cdots e_k$ such that

$$e_1 = (v_1, v_2), e_2 = (v_2, v_3), \dots, e_k = (v_k, v_{k+1})$$

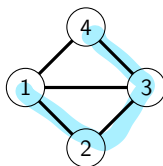
is a *path* from v_1 to v_{k+1} .

Notation: $v_1 v_2 \cdots v_{k+1}$

Path classification

- A *path* is *simple* if it does not pass through the same *vertex* twice (with the exception of the starting node, when it coincides with the end node)
- A *cycle* is a *SIMPLE* path with $v_{k+1} = v_1$ and crossing at least 3 distinct nodes

Connectivity in *undirected* graphs



- The paths 1234 and 12341 are simple
- The path 1231 is a cycle
- The path 121 is not a cycle. Also 12312341 is not a cycle

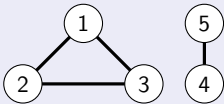
1 2 3 4

Proof that $G = (\{u\}, \emptyset)$ is connected. G connected if $\exists u, v \in V$ and $u \neq v \Rightarrow u$ connected to v

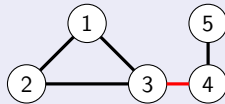
Connectivity and completeness

- A node v_2 is connected to v_1 if there is a path from v_1 to v_2 $\hookrightarrow$ true if $V = \{u\}$
- A graph is *connected* if all pairs of distinct vertices are connected $\downarrow$ therefore the implication is true $\hookrightarrow G$ is connected
 - A graph $G = (V, E)$ is *complete* if, for all pairs of distinct vertices, there is an edge connecting them

Disconnected graph



Connected graph



Remark

H_1 $H_2 \rightarrow H_1$ and H_2 are connected components

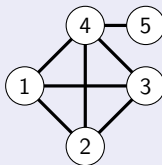
If a graph is *disconnected*, then it is *composed* of *multiple connected subgraphs*, called *connected components*

Connectivity in *undirected* graphs

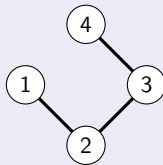
Tree

A tree of the undirected graph $G = (V, E)$ is a **connected acyclic** subgraph. A tree is **spanning** if it contains n nodes, where $n = |V|$.

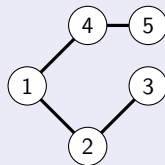
Graph G



Tree



Spanning tree

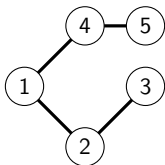


Connectivity in *undirected* graphs

Tree theorem

Let T be an undirected graph. The following conditions are equivalent:

- T is a tree
- T is connected and has n nodes and $n - 1$ edges
- Every pair of nodes of T is connected by a unique simple path



Directed graphs

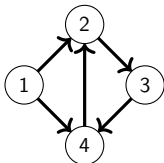
Digraphs: neighbors of a node

Let $G = (V, E)$ be a digraph and $(u, v) \in E$

- u is an *in-neighbor* of v and v is an *out-neighbor* of u
- $\mathcal{N}^{in}(u)$ and $\mathcal{N}^{out}(u)$ are the sets of in/out neighbors of u
- the in-degree and out-degree of u are defined as

$$d^{in}(u) = \sum_{j \in \mathcal{N}^{in}(u)} w_{ju} \quad d^{out}(u) = \sum_{j \in \mathcal{N}^{out}(u)} w_{uj}$$

$\underbrace{j \in \mathcal{N}^{in}(u)}_{\hookrightarrow \{j : (j, u) \in E\}} \quad \underbrace{j \in \mathcal{N}^{out}(u)}_{\hookrightarrow \{j : (u, j) \in E\}}$



$$d^{in}(2) = 2 \quad \mathcal{N}^{in}(2) = \{1, 4\}$$
$$d^{out}(2) = 1 \quad \mathcal{N}^{out}(2) = \{3\}$$

Digraphs: neighbors of a subset of nodes

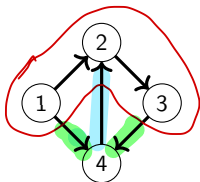
Let $G = (V, E)$ be a digraph and $S \subseteq V$. The set of

- out-neighbors of S is

$$\mathcal{N}^{out}(S) = \{j \in V \setminus S : (i, j) \in E \text{ for some } i \in S\}$$

- in-neighbors of S is $\mathcal{N}^{in}(S) = \{i \in V \setminus S : (i, j) \in E \text{ for some } j \in S\}$

The set $V \setminus S$ is the difference of sets V and S , collecting all vertices in V that are not in S



$$\mathcal{N}^{out}(\{1, 2, 3\}) = \{4\}$$

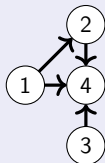
$$\mathcal{N}^{in}(\{1, 2, 3\}) = \{4\}$$

Directed graphs: subgraphs

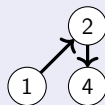
Subgraph

The digraph $H = (U, F)$ is a *subgraph* of the digraph $G = (V, E)$ if $U \subseteq V$, $F \subseteq E$ and edges in F connect only nodes in U

Graph $G = (V, E)$



Graph $H = (U, F)$



$H = (U, F)$ is a subgraph of G because $U = \{1, 2, 4\} \subseteq V$,
 $F = \{(1, 2), (2, 4)\} \subseteq E$ and edges in F connects only nodes in U .

Spanning subgraph

A subgraph is **spanning** if its **node set is V**

Connectivity in *digraphs*

Sources and sinks

- A **source** is a node v with no in-neighbors
- A **sink** is a node v with no out-neighbors



Path and cycles

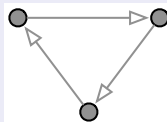
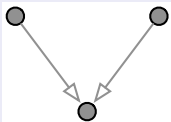
- A path $v_1 \dots v_k$ is defined as for undirected graphs
- As for undirected graphs, a path is *simple* if it does not pass through the **same vertex twice** (with the **exception of the starting node**, when it coincides with the end node)
- A *cycle* is a **SIMPLE path starting and ending in the same node**
- **Remember:**
 - ▶ self-loops are (v, v) are allowed in digraphs
 - ▶ (v, v) and $(v, u)(u, v)$ are cycles in digraphs



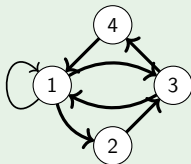
Connectivity in *digraphs*

Remark

Every **acyclic** digraph has **always** at least one **source** and one **sink**



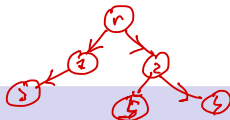
Example - simple paths and cycles



- The path 1234 is simple
- The paths 1231, 131, and 11 are cycles

Connectivity in *digraphs*

Directed tree



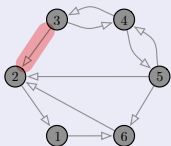
A directed tree is an acyclic digraph where there is a node r (called the root) such that any other node $v \in V$ can be reached from r through one and only one path

Weak/strong connectivity

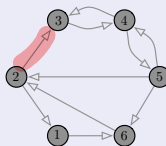
- G is *strongly connected* if for any $u, v \in V$, $u \neq v$ there is a path from u to v
- G is *weakly connected* if its undirected version is connected

→ the graph obtained by neglecting the direction of the edges

Weakly connected G



Strongly connected G



Weakly connected
Strongly connected

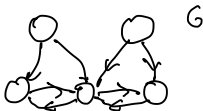
Examples

Connectivity in digraphs



Directed tree

A directed tree is an acyclic digraph where there is a node r (called the root) such that any other node $v \in V$ can be reached from r through one and only one path.



Is G a directed tree? NO: no root, multiple paths...

In a directed tree

• Can we have two roots? NO

By contradiction, assume r_1 and r_2 are roots



• Is the root a source? YES

By contradiction, assume r is a root and not a source

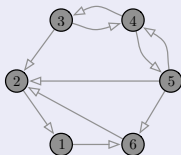


Connectivity in *digraphs*

Global reachable node and directed spanning tree

- G has a *globally reachable node* if there is $g \in V$ connected to any other node $v \in V$ (i.e. there is a path from v to g)
- G has a *directed spanning tree* if it contains a directed tree comprising all nodes in V

G with a globally reachable node (node 2)



Connectivity in *digraphs*

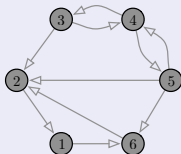
Reverse graph

Let $G = (V, E)$ be a digraph. Its reverse graph $G^{rev}(V, E^{rev})$ is given by the edge set $E^{rev} = \{(i, j) \text{ if } (j, i) \in E\}$

Remark

G has a globally reachable node, if and only if G^{rev} includes a directed spanning tree

G with a globally reachable node (node 2)



Take home messages and open problems

- Graph theory allows one to easily model the topology of a control network
- Graphs define connectivity properties
- Any algebraic characterization of graph connectivity properties ?
- How graph connectivity relates to the achievement of consensus in the examples ?