Lecture 4

Packet dropouts

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Packet dropout

Causes: node failures or message collisions

- transmission-retry mechanisms: retransmit for a limited time
- for real-time feedback control it might beneficial that the controller discards retransmission of sensor measurements if new ones are available

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Problem

How dropouts affect stability of an NCS ?

Models of dropouts

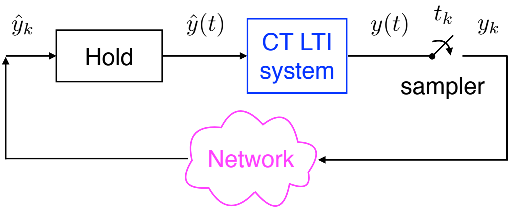
- Deterministic
  - average dropout rate
  - worst case bound on n° of consecutive dropouts (not in this class)
- Stochastic
  - Bernoulli process
  - Finite-state Markov chains for correlated dropouts (not in this class)

Outline

- Models of NCS with packet dropout
- Stability under deterministic dropout
  - Estimation of the maximal admissible dropout rate
- Stability under stochastic packet dropout

NCS model

Collocated control



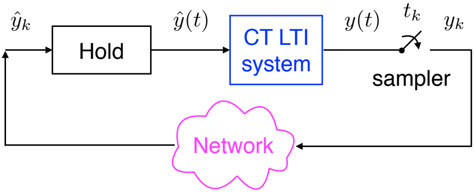
SISO CT LTI system

$$\begin{cases} \dot{x} = Ax + B\hat{y} \\ y = Cx \end{cases}$$

Sampling times $\{t_k, k \in \mathbb{N}\}$, $T_k = t_{k+1} - t_k$

NCS dropout model

Collocated control



Network model (packet dropout)

$$\hat{y}_k = \theta_k y_k + (1-\theta_k)\hat{y}_{k-1} = \begin{cases} y_k & \theta_k = 1 \text{ (no dropout)} \\ \hat{y}_{k-1} & \theta_k = 0 \text{ (dropout)} \end{cases}$$

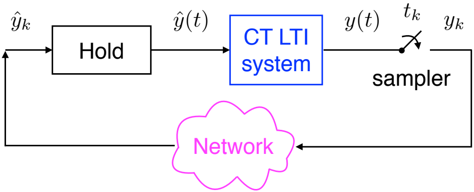
The "^" is important as it denotes the last received measurement (could be y_{k-100} at time k...)

Remark

$\hat{y}_k$ is <u>not</u> set to zero if $\theta_k = 0$

NCS dropout model

Collocated control



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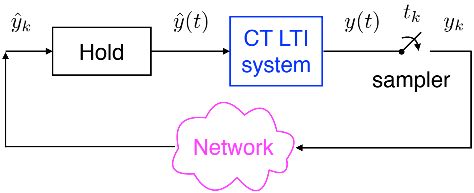
The "^" is important as it denotes the last received measurement (could be y_{k-100} at time k...)

Standing assumptions (for simplicity)

Uniform sampling ($T_k = T$), constant network delay ($\tau_k = \tau$) and $\tau < T$

NCS model: dropout+network delay

Collocated control



Model of system input with delayed transmission

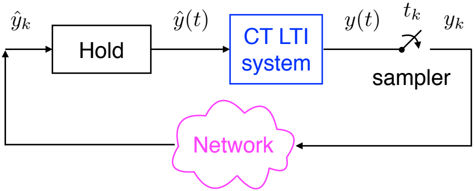
$$\hat{y}(t) = \begin{cases} \hat{y}_{k-1} & t \in [t_k, t_k + \tau) \\ \hat{y}_k & t \in [t_k + \tau, t_{k+1}) \end{cases}$$

Remark

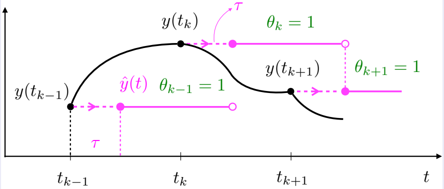
It makes sense to consider simultaneously packet dropout and delay, as the latter has a non trivial effect on stability

NCS model: dropout+network delay

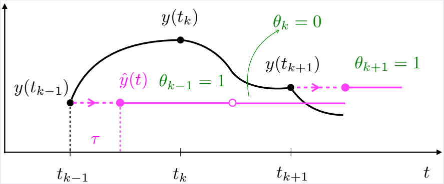
Collocated control



No drop



One drop



NCS model: dropout+network delay

Collocated control

Define the augmented state $z_k = \begin{bmatrix} x_k \\ \hat{y}_{k-1} \end{bmatrix}$

Discrete-time (DT) NCS model

$$z_{k+1} = \psi_{\theta_k} z_k$$

$$\psi_\theta = \begin{bmatrix} e^{AT} + \theta\Gamma(T-\tau)BC & e^{A(T-\tau)}\Gamma(\tau)B + (1-\theta)\Gamma(T-\tau)B \\ \theta C & (1-\theta)I \end{bmatrix}$$

where $\Gamma(s) = \int_0^s e^{At} dt$

- $\theta = 1$ (transmission): same model we have seen for analyzing delays
- $\theta = 0$ (packet loss)

Remark

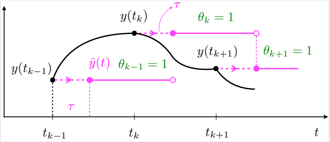
The NCS is a switched system, i.e. a system with a discrete-valued input deciding the active model within a finite set of possible ones (2 in our case)

Derivation of the NCS model

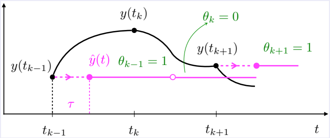
$$z_{k+1} = \psi_{\theta_k} z_k \qquad z_k = \begin{bmatrix} x_k \\ \hat{y}_{k-1} \end{bmatrix}$$

$$\psi_\theta = \begin{bmatrix} e^{AT} + \theta\Gamma(T-\tau)BC & e^{A(T-\tau)}\Gamma(\tau)B + (1-\theta)\Gamma(T-\tau)B \\ \theta C & (1-\theta)I \end{bmatrix} \qquad (1)$$

No drop



One drop



The second row of (1) is the packet-drop model. First row of (1): for $\tau < T$ we have seen previously (lectures on delays)

$$x_{k+1} = e^{AT}x_k + e^{A(T-\tau)}\Gamma(\tau)B\hat{y}_{k-1} + \Gamma(T-\tau)B\hat{y}_k$$

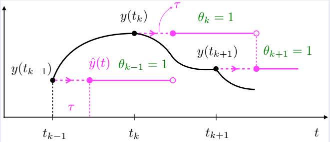
Substitute the packet drop model $\hat{y}_k = \theta_k y_k + (1-\theta_k)\hat{y}_{k-1}$, and obtain the result. More in details...

Derivation of the NCS model

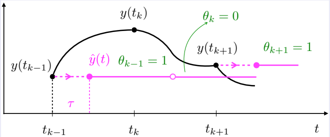
$$z_{k+1} = \psi_{\theta_k} z_k \qquad z_k = \begin{bmatrix} x_k \\ \hat{y}_{k-1} \end{bmatrix}$$

$$\psi_\theta = \begin{bmatrix} e^{AT} + \theta\Gamma(T-\tau)BC & e^{A(T-\tau)}\Gamma(\tau)B + (1-\theta)\Gamma(T-\tau)B \\ \theta C & (1-\theta)I \end{bmatrix} \tag{1}$$

No drop



One drop



$$\begin{aligned} x_{k+1} &= e^{AT}x_k + e^{A(T-\tau)}\Gamma(\tau)B\hat{y}_{k-1} + \Gamma(T-\tau)B\theta_k \overbrace{y_k}^{=Cx_k} + \\ &+ \Gamma(T-\tau)B(1-\theta_k)\hat{y}_{k-1} = \\ &= (e^{AT} + \theta_k\Gamma(T-\tau)BC)x_k + (e^{A(T-\tau)}\Gamma(\tau)B + \\ &+ \Gamma(T-\tau)B(1-\theta_k))\hat{y}_{k-1} \end{aligned}$$

Stability of NCSs under deterministic packet dropout

Deterministic dropout model

Definition

The asymptotic packet dropout rate $r \in [0,1]$ is given by

$$r = \lim_{N\to\infty} \frac{1}{N} \sum_{k=k_0}^{k_0+N+1} (1-\theta_k), \qquad \forall k_0 \in \mathbb{N} \tag{2}$$

Note that r is independent of k_0

Standing assumption

For all sequences θ_k, r exists.

Problem

How much r affects NCS stability?

Deterministic dropout: stability test

Theorem 1

Assume there is $P = P^T > 0$ and scalars α, α_0, α_1 such that

$$\alpha_0^r \alpha_1^{1-r} > \alpha > 1 \tag{3}$$

$$\psi_0(T,\tau)^T P \psi_0(T,\tau) \leq \alpha_0^{-2} P \tag{4}$$

$$\psi_1(T,\tau)^T P \psi_1(T,\tau) \leq \alpha_1^{-2} P \tag{5}$$

Then, the NCS is exponentially stable with rate $\log \frac{1}{\alpha}$

Remarks

1. (4)-(5) are LMIs for fixed α_0, α_1 (sufficient condition only)
2. (3)-(4)-(5) are bilinear matrix inequalities in P, α, α_0, α_1 $\Rightarrow$ Idea (approximate solution): grid the region of the (α_0, α_1)-plane verifying $\alpha_0^r \alpha_1^{1-r} > 1$ and solve the LMIs (4)-(5)
3. Once α_0, α_1 are fixed, α can be chosen to fulfill (3)
4. $z^T P z$ is a common Lyapunov function for the switched system

Proof of Theorem 1 (check @ home)

Recall

"Exponentially stable with rate $\log \frac{1}{\alpha}$": $\exists c > 0$, $\frac{1}{\alpha} \in (0,1)$ $||x_k|| \leq c||x_0|| \left(\frac{1}{\alpha}\right)^k$

Proof

Let $V(z) = z^T P z$ which is > 0 for $z \neq 0$. Then $V(z_k) = z_k^T P z_k = z_{k-1}^T \left(\psi_{\theta_{k-1}}^T(T,\tau) P \psi_{\theta_{k-1}}(T,\tau)\right) z_{k-1}$. From (4) and (5), one has

$$
\begin{aligned}
V(z_k) &\leq \alpha_{\theta_{k-1}}^{-2} V(z_{k-1}) \leq \alpha_{\theta_{k-1}}^{-2} \alpha_{\theta_{k-2}}^{-2} V(z_{k-2}) \leq \\
&\leq \underbrace{\alpha_{\theta_{k-1}}^{-2} \cdot \dots \cdot \alpha_{\theta_0}^{-2}}_{(a)} V(z_0)
\end{aligned}
$$

In the product (a), for $k \to \infty$, α_0^{-2} appears rk times and α_1^{-2} appears $(1-r)k$ times. Hence, for $k \to \infty$

$$
\begin{aligned}
V(z_k) &\leq (\alpha_0^{-2})^{rk} (\alpha_1^{-2})^{(1-r)k} V(z_0) = \\
&= \left(\alpha_0^r \alpha_1^{(1-r)}\right)^{-2k} V(z_0) \leq \alpha^{-2k} V(z_0)
\end{aligned}
\tag{6}
$$

Proof of Theorem 1 (check @ home)

Let us now prove ES in the usual way.
Since $P > 0$, there are $\beta, \gamma > 0$ such that $\beta I < P \leq \gamma I$, i.e. $\beta||z||^2 \leq V(z) \leq \gamma||z||^2$.
Then, (6) gives

$$\beta||z_k||^2 \leq V(z_k) \leq \alpha^{-2k} V(z_0) \leq \alpha^{-2k}\gamma||z_0||^2$$

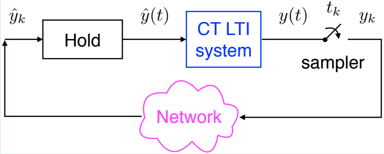
Hence

$$||z_k||^2 \leq \alpha^{-2k}\frac{\gamma}{\beta}||z_0||^2$$

i.e.

$$||z_k|| \leq \alpha^{-k}||z_0||c, \quad \text{with } c = \sqrt{\frac{\gamma}{\beta}}$$

Example - collocated control, packet loss



- Uniform sampling: $T = 0.1$
- No delay, i.e. $\tau = 0$

System

$$\begin{cases} \dot{x} = 0.2x + u \\ y = -18x \end{cases}$$

Hold

$$\hat{y}_k = \begin{cases} y(t_k) & \theta_k = 1 \\ \hat{y}_{k-1} & \theta_k = 0 \end{cases}$$

$$e^{AT} = 1.0202, \qquad \Gamma(T) = \int_0^T e^{As} ds = 0.1010$$

Build the DT NCS model $z_{k+1} = \psi_{\theta_k} z_k$ with $z_k = \begin{bmatrix} x_k \\ \hat{y}_{k-1} \end{bmatrix}$

Example - collocated control, packet loss

$$\psi_\theta = \begin{bmatrix} e^{AT} + \theta\Gamma(T)BC & e^{A(T)}\underbrace{\Gamma(0)}_{=0}B + (1-\theta)\Gamma(T)B \\ \theta C & (1-\theta)I \end{bmatrix}$$

$$\psi_0 = \begin{bmatrix} 1.0202 & 0.1010 \\ 0 & 1 \end{bmatrix} \qquad \psi_1 = \begin{bmatrix} \underbrace{1.0202 - 1.8181}_{-0.7979} & 0 \\ -18 & 0 \end{bmatrix}$$

$\mathrm{Spec}(\psi_0) = \{1.0202, 1\} \rightarrow$ unstable
$\mathrm{Spec}(\psi_1) = \{-0.7978, 0\} \rightarrow$ AS

Example - application of the stability theorem

Find $\alpha_0^r \cdot \alpha_1^{1-r} > \alpha > 1$ such that, for some $P = P^T > 0$

$$\psi_0^T P \psi_0 \leq \alpha_0^{-2} P \tag{7}$$

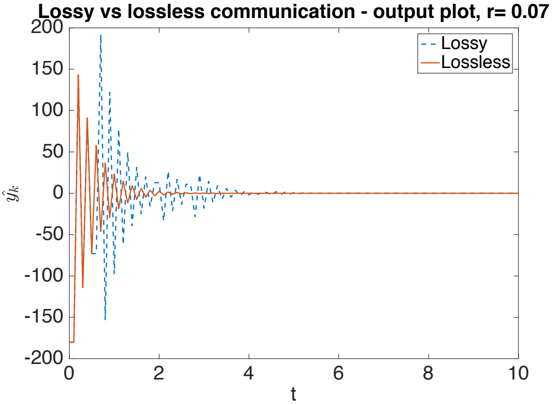
$$\psi_1^T P \psi_1 \leq \alpha_1^{-2} P \tag{8}$$

Remark

- ψ_0 is unstable $\Rightarrow \alpha_0 \leq 1$ (hence $\alpha_0^{-2} \geq 1$)
- $P = 0$ always verifies (7) and (8). It is important to check that $P > 0$ and not only that $P \geq 0$.

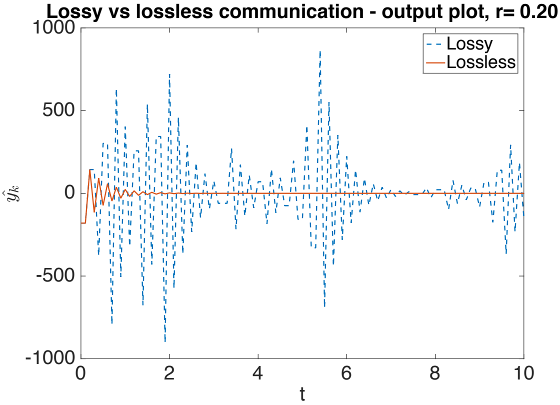
Application of the stability theorem

- For $r = 0.07$, (7) and (8) are feasible for $\alpha_0 = 0.12$ and $\alpha_1 = 1.89$ $(\alpha_0^r \cdot \alpha_1^{1-r} = 1.5575)$. Then the NCS is AS with rate $\log \frac{1}{\alpha}$ where $\frac{1}{\alpha} > \frac{1}{1.5575} = 0.6421$. This means $\|x_k\| \le c\|x_0\| \left(\frac{1}{\alpha}\right)^k$, for a suitable constant $c > 0$.



Application of the stability theorem

- LMIs are feasible, for suitable α_0, α_1, also for $r = 0.20$



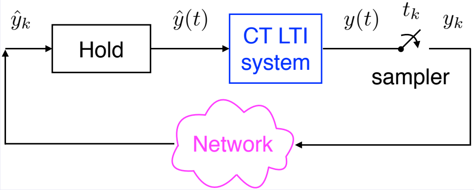
Problem

How to avoid trial-and-error for estimating the maximal drop rate that can be tolerated ?

Deterministic dropouts: estimation of the maximal admissible packet drop rate

Reference setting

Collocated control



At which maximal rate one can drop packets while preserving exponential stability?

Before answering, let us consider the system dynamics in the two extreme cases $\theta_k = 0$ and $\theta_k = 1$, $k = 0, 1, 2 \dots$

Case $\theta_k = 1,\ k = 0, 1, 2 \ldots$ (no dropout)

Recall the definition of the augmented state $z_k = \begin{bmatrix} x_k \\ \hat{y}_{k-1} \end{bmatrix}$

Discrete-time (DT) NCS model

$$z_{k+1} = \psi_1 z_k$$

$$\psi_1 = \begin{bmatrix} e^{AT} + \Gamma(T-\tau)BC & e^{A(T-\tau)}\Gamma(\tau)B \\ C & 0 \end{bmatrix}$$

Stability in presence of delay has already been studied!

Assumption

ψ_1 is Schur Stable, i.e. $\rho(\psi_1) < 1$ where $\rho(\cdot)$ is the spectral radius

Recall

For $M \in \mathbb{R}^{n\times n}$, the spectral radius is $\rho(M) = \max\{|\lambda_i|, i = 1, \ldots, n\}$ where λ_i are the eigenvalues of M

Case $\theta_k = 1,\ k = 0, 1, 2 \ldots$ (no dropout)

Recall the definition of the augmented state $z_k = \begin{bmatrix} x_k \\ \hat{y}_{k-1} \end{bmatrix}$

Discrete-time (DT) NCS model

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Stability in presence of delay has already been studied!

Assumption

ψ_1 is Schur Stable, i.e. $\rho(\psi_1) < 1$ where $\rho(\cdot)$ is the spectral radius

This assumption implies that, if $r = 0$, then the NCS is exponentially stable

Case $\theta_k = 0,\ k = 0, 1, 2 \dots$ (dropout)

Discrete-time (DT) NCS model

$$z_{k+1} = \psi_0 z_k$$

$$\psi_0 = \begin{bmatrix} e^{AT} & (e^{A(T-\tau)}\Gamma(\tau) + \Gamma(T-\tau))B \\ 0 & I \end{bmatrix}$$

Since ψ_0 is block-triangular, its spectral radius is

- 1 if e^{AT} is Schur stable, i.e. if the LTI system is open-loop stable
- $\rho(e^{AT})$ if $\rho(e^{AT}) > 1$

This implies that, if $r = 1$, the NCS is not asymptotically stable

Summary

For all NCSs that are asymptotically stable in presence of the delay τ

- if $r = 0$ the NCS is asymptotically stable
- if $r = 1$ the NCS is NOT asymptotically stable

Intuition

One expects that there is a maximum asymptotic rate $r_{max} \in [0, 1)$ such that $r < r_{max}$ guarantees asymptotic stability

Problem

How to estimate r_{max}?

Idea

Build on the LMI-based stability theorem previously seen ..

Summary

Theorem 2 (estimation of the maximal dropout rate)

Assume there is $\beta_0 \geq 1$, $\beta_1 < 1$ and $P = P^T > 0$ such that

$$\psi_0(T,\tau)^T P \psi_0(T,\tau) \leq \beta_0 P \tag{9}$$

$$\psi_1(T,\tau)^T P \psi_1(T,\tau) \leq \beta_1 P \tag{10}$$

Then, the NCS is exponentially stable for all $r < \bar{r}$ where

$$\bar{r} = \frac{1}{1 - \frac{\gamma_0}{\gamma_1}} \tag{11}$$

$$\gamma_0 = \log(\beta_0) \qquad \gamma_1 = \log(\beta_1)$$

Proof of Theorem 2 (check @ home)

For given $\beta_0 \geq 1$ and $\beta_1 < 1$ we look for values of r that verify the inequality (12) of Theorem 1, here copied for convenience

Theorem 1

Assume there is $P = P^T > 0$ and scalars α, α_0, α_1 such that

$$\alpha_0^r \alpha_1^{1-r} > \alpha > 1 \tag{12}$$

$$\psi_0(T,\tau)^T P \psi_0(T,\tau) \leq \alpha_0^{-2} P \tag{13}$$

$$\psi_1(T,\tau)^T P \psi_1(T,\tau) \leq \alpha_1^{-2} P \tag{14}$$

Then, the NCS is exponentially stable with rate $\log \frac{1}{\alpha}$

Since $\beta_0 = \alpha_0^{-2}$ and $\beta_1 = \alpha_1^{-2}$, the inequality $\alpha_0^r \alpha_1^{1-r} > 1$ gives $\beta_0^{-\frac{1}{2}r} \beta_1^{-\frac{1}{2}(1-r)} > 1$. Taking the log of both sides

$$-\frac{1}{2} r \log(\beta_0) - \frac{1}{2}(1-r)\log(\beta_1) > 0 \Rightarrow r \underbrace{\left(\log(\beta_1) - \log(\beta_0)\right)}_{<0} > \underbrace{\log(\beta_1)}_{<0}$$

which is possible if

$$r < \frac{1}{1 - \frac{\log(\beta_0)}{\log(\beta_1)}}$$

The inequalities (9) and (10) imply that also (13) and (14) are fulfilled. In view of Theorem 1 the proof is complete.

Remarks

- To verify $\psi_0(T,\tau)^T P \psi_0(T,\tau) \leq \beta_0 P$ it must hold that (proof not shown)

$$\beta_0 \geq \rho^2(\psi_0(T,\tau)) \geq 1$$

- To verify $\psi_1(T,\tau)^T P \psi_1(T,\tau) \leq \beta_1 P$ it must hold that

$$\beta_1 \geq \rho^2(\psi_1(T,\tau))$$

  Since $\rho^2(\psi_1(T,\tau)) < 1$ it might be possible to have $\beta_1 < 1$
- Recall that

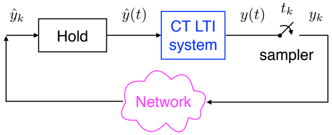
$$\bar{r} = \frac{1}{1 - \frac{\gamma_0}{\gamma_1}}, \quad \gamma_0 = \log(\beta_0) \geq 0 \qquad \gamma_1 = \log(\beta_1) < 0$$

  which implies $\bar{r} < 1$

To maximize $\bar{r}$, one should choose,

- β_0 as close as possible to 1. If e^{AT} is Schur stable, $\rho^2(\psi_0(T,\tau)) = 1 \Rightarrow \beta_0 = 1$ is feasible.
- β_1 as small as possible

Example



- $A = \begin{bmatrix} 0.1 & 0.098 \\ 0 & -1 \end{bmatrix}$ unstable
- $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$
- $C = \begin{bmatrix} -12.45 & -1.11 \end{bmatrix}$

$\mathrm{Spec}(A + BC) = \{-1, -1.01\} \Rightarrow A + BC$ Hurwitz

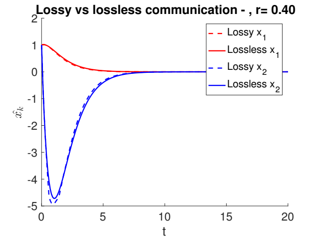
- LMIs in Theorem 2 are feasible for $\beta_0 = 4$ and $\beta_1 = 0.2325$.

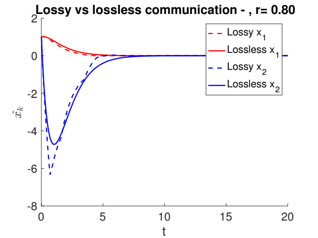
$$\gamma_1 = \log(\beta_1) = -1.4589, \ \gamma_0 = \log(\beta_0) = 1.3863,$$

$$\bar{r} = \frac{1}{1 - (\gamma_0/\gamma_1)} = 0.5124$$

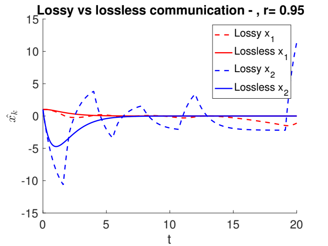
Example

Plots of the states for different drop rates





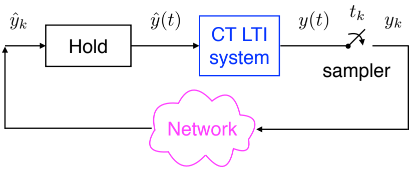
How much conservative is $\bar{r} = 0.5124$?



Stochastic dropouts

Stochastic dropouts

NCS scheme - Collocated control - Review of the setup



LTI system

$$\begin{cases} \dot{x} = Ax + B\hat{y} \\ y = Cx \end{cases}$$

Assumptions

- SISO system
- Uniform sampling period T and network delay $\tau < T$

Network model (packet dropout)

$$\hat{y}_k = \theta_k y_k + (1 - \theta_k)\hat{y}_{k-1} = \begin{cases} y_k & \text{if } \theta_k = 1 \text{ (no dropout)} \\ \hat{y}_{k-1} & \text{if } \theta_k = 0 \text{ (dropout)} \end{cases}$$

Stochastic dropouts

NCS scheme - Review of the setup

NCS model

Setting $z_k = \begin{bmatrix} x_k \\ \hat{y}_{k-1} \end{bmatrix}$, one has

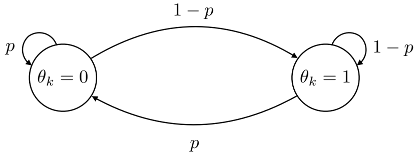
$$z_{k+1} = \psi_{\theta_k} z_k \tag{15}$$

$$\psi_\theta = \begin{bmatrix} e^{AT} + \theta\Gamma(T-\tau)BC & e^{A(T-\tau)}\Gamma(\tau)B + (1-\theta)\Gamma(T-\tau)B \\ \theta C & (1-\theta)I \end{bmatrix}$$

where $\Gamma(s) = \int_0^s e^{At} dt$

- $\theta = 1$ (transmission): same model we have seen for analyzing delays
- $\theta = 0$ (packet loss)

Stochastic dropout model

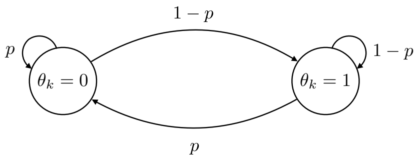


- Markov chain with 2 states (dropout state: $\theta_k = 0$)
- $p \in [0, 1)$ probability of dropout, uniform in time (can be a strong assumption, e.g. for wireless networks)
- θ_k is a random variable with Bernoulli distribution
  - Recall:

$$\mathbb{E}[\theta_k] = \mathrm{Prob}(\theta_k = 1) = 1 - p$$
$$\mathrm{Var}[\theta_k] = p(1 - p)$$

The NCS becomes a "Markovian Jump Linear System" (coupling between discrete and continuous states, where discrete states obey to a Markov chain)

Stochastic dropout model



Definition

The NCS $z_{k+1} = \psi_k z_k$ is *mean-square stable* if, for every initial state x_0

$$\lim_{k\to\infty} \mathbb{E}[x_k] = 0 \tag{16}$$

and

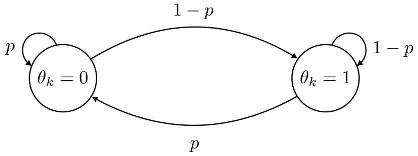
$$\lim_{k\to\infty} \mathbb{E}[x_k x_k^T] = 0 \tag{17}$$

Recall

$$\mathbb{E}[x_k x_k^T] = \mathrm{Var}(x_k) + \mathbb{E}[x_k]\mathbb{E}[x_k]^T$$

Then (16) + (17) $\Rightarrow \mathrm{Var}(x_k) \to 0$ as $k \to +\infty$

Stochastic dropout model



Problem:

How to analyze mean-square stability ?

NCS as an average system with stochastic uncertainty

Goal

Rewrite model (15) in a more meaningful form

Define $\Delta_k = \frac{\theta_k}{1-p} - 1 \in \{-1, \frac{p}{1-p}\}$

- Stochastic perturbation with mean $\mathbb{E}[\Delta_k] = \frac{\mathbb{E}[\theta_k]}{1-p} - 1 = 0$, since $\mathbb{E}[\theta_k] = \mathrm{Prob}(\theta_k = 1) = 1 - p$
- Variance $\sigma^2 = \mathbb{E}[\Delta_k^2] = \frac{p}{1-p}$

Trick: $\theta_k = (1-p)(1+\Delta_k)$

NCS as an average system with stochastic uncertainty

Proposition

The model (15) is equivalent to

$$\Sigma : \begin{cases} z_{k+1} = \bar{A} z_k + \bar{B} \hat{v}_k & (18) \\ v_k = \bar{C} z_k & (19) \end{cases}$$

$$\hat{v}_k = \Delta_k v_k \tag{20}$$

where

$$\bar{A} = \begin{bmatrix} e^{AT} + (1-p)\Gamma(T-\tau)BC & e^{A(T-\tau)}\Gamma(\tau)B + p\Gamma(T-\tau)B \\ (1-p)C & pI \end{bmatrix},$$

$\bar{B} = \begin{bmatrix} (1-p)\Gamma(T-\tau)B \\ (1-p)I \end{bmatrix}$ and $\bar{C} = \begin{bmatrix} C & -I \end{bmatrix}$. Moreover, $(\bar{A}, \bar{B})$ is termed the average NCS model.

Proof

From (15), i.e.

$$z_k = \begin{bmatrix} x_k \\ \hat{y}_{k-1} \end{bmatrix} \qquad z_{k+1} = \psi_{\theta_k} z_k$$

$$\psi_\theta = \begin{bmatrix} e^{AT} + \theta\Gamma(T-\tau)BC & e^{A(T-\tau)}\Gamma(\tau)B + (1-\theta)\Gamma(T-\tau)B \\ \theta C & (1-\theta) \end{bmatrix}$$

by using $\theta_k = (1-p)(1+\Delta_k) = (1-p) + \Delta_k(1-p)$, one has

$$\begin{aligned} x_{k+1} &= \left(e^{AT} + (1-p)\Gamma(T-\tau)BC + (1-p)\Delta_k\Gamma(T-\tau)BC\right)x_k + \\ &+ \left(e^{A(T-\tau)}\Gamma(\tau)B + p\Gamma(T-\tau)B\right)\hat{y}_{k-1} - \Delta_k(1-p)\Gamma(T-\tau)B\hat{y}_{k-1} = \\ &= \left(e^{AT} + (1-p)\Gamma(T-\tau)BC\right)x_k + \left(e^{A(T-\tau)}\Gamma(\tau)B + p\Gamma(T-\tau)B\right)\hat{y}_{k-1} + \\ &+ (1-p)\Gamma(T-\tau)B(\underbrace{Cx_k - \hat{y}_{k-1}}_{\hat{v}_k})\Delta_k \end{aligned}$$

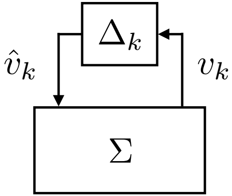
So we obtained the blocks in the first row of $\bar{A}$ and $\bar{B}$. The blocks in the second row can be obtained is a similar way.

Interpretation of the NCS model

$$\Sigma : \begin{cases} z_{k+1} = \bar{A} z_k + \bar{B} \hat{v}_k \\ v_k = \bar{C} z_k \end{cases}$$

$$\hat{v}_k = \Delta_k v_k$$

Representation of the NCS



Remark

- Σ is a nominal deterministic system with stochastic perturbation Δ_k
- This representation allows one to cast the stability problem into a robust stability problem

Mean-square stability of the NCS

Theorem 4

Assume that $\bar{A}$ is Schur. Then, the NCS (18)-(20) is mean-square stable if and only if there is $P = P^T > 0$ and a scalar $\alpha > 0$ such that

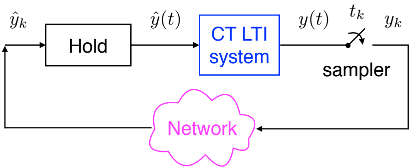
$$\bar{A}P\bar{A}^T + \alpha\bar{B}\bar{B}^T < P \tag{21}$$

$$\frac{p}{1-p}\bar{C}P\bar{C}^T < \alpha \tag{22}$$

Remarks

- Necessary and sufficient condition !
- LMIs in $P > 0$ and $\alpha > 0$!
- (21) $\Leftrightarrow \bar{A}P\bar{A}^T - P < -\alpha\bar{B}\bar{B}^T \Leftrightarrow V(z) = z^T Pz$ is a Lyapunov function certifying that $\bar{A}^T$ is Schur $\Leftrightarrow \bar{A}$ is Schur.
- The average NCS must be AS.

Example



LTI system

$$\dot{x} = 0.2x + u$$
$$y = -18x$$

Sampling period $T = 0.1$ and delay $\tau = 0$

Is the NCS mean-square stable for the packet loss probability $p = 0.03$?

Remark

Unstable open-loop system $\rightarrow$ packet drop is critical

Solution
Write the NCS as

$$\begin{cases} z_{k+1} = \bar{A} z_k + \bar{B} \hat{v}_k \\ v_k = \begin{bmatrix} C & -I \end{bmatrix} z_k \\ \hat{v} = \Delta_k v_k \end{cases}$$

Example

Since

$$e^{AT} = 1.0202, \qquad \Gamma(T) = \int_0^T e^{As} ds = 0.1010, \qquad \Gamma(\tau) = 0,$$

one has

$$\bar{A} = \begin{bmatrix} 1.0202 + (1-p)(-1.8181) & p0.1010 \\ (1-p)(-18) & p \end{bmatrix}$$

and, since $p = 0.03$

$$\begin{bmatrix} -0.7434 & 0.003 \\ -17.46 & 0.03 \end{bmatrix} \rightarrow \mathrm{Spec}(\bar{A}) = \{-0.6675, -0.0458\}$$

$\bar{A}$ is Schur: the Theorem can be applied. One also has

$$\bar{B} = \begin{bmatrix} (1-p)0.1010 \\ (1-p) \end{bmatrix} = \begin{bmatrix} 0.098 \\ 0.97 \end{bmatrix}, \qquad \bar{C} = \begin{bmatrix} -18 & -1 \end{bmatrix}$$

Example

LMIs in the unknowns $P \in \mathbb{R}^{2\times 2}$ and $\alpha \in \mathbb{R}$

$$
\begin{aligned}
P^T = P > 0,\ \alpha > 0 \\
\bar{A}P\bar{A}^T + \bar{B}\alpha\bar{B}^T < P \\
\frac{p}{1-p}\bar{C}P\bar{C}^T < \alpha
\end{aligned}
$$

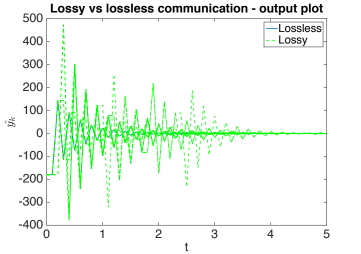
From MatLab + Yalmip:

$$
\alpha = 0.5910, \qquad P = \begin{bmatrix} 0.012 & 0.1916 \\ 0.1916 & 4.6474 \end{bmatrix}
$$

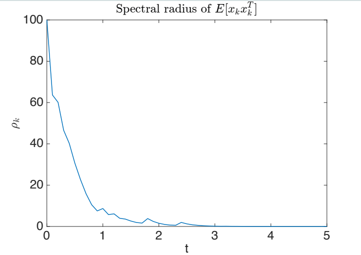
$\Rightarrow$ the NCS is mean-square stable

Example

Plot of 100 simulations



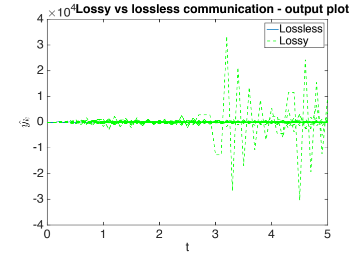
Sample estimate of $\rho(\mathbb{E}[x_k x_k^T])$



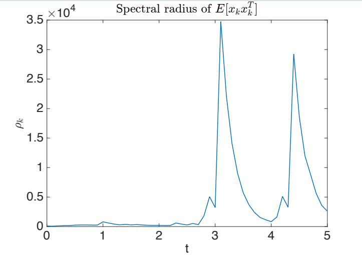
Example

For $p = 0.1$ LMIs are infeasible

Plot of 100 simulations



Sample estimate of $\rho(\mathbb{E}[x_k x_k^T])$



Take home messages

- Packet dropout can compromise NCS stability, especially if the system under control is unstable
- Stability tests based on the LMIs exist
  - For deterministic dropouts with finite asymptotic loss rate
  - For stochastic dropouts with uniform loss probability