Lecture 3-b

Data-rate limitations and quantization

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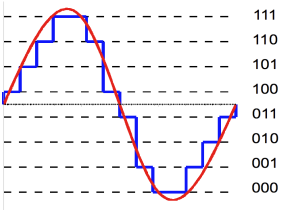
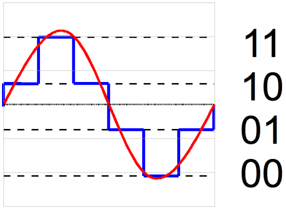
Quantization

A packet can contain only finitely many bits N_B

- *Quantization*: real-valued vectors (e.g. the control variable) must be coded into N_B bits before being transmitted
- Small packets $\rightarrow$ non-negligible approximation errors
- Significant constraint for control networks with low bandwidth or battery-driven sensors connected through wireless networks and aiming at minimizing the communication energy



["Quantization", Wikipedia]



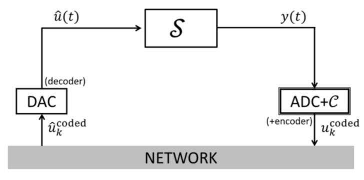
Outline

Study the effect of quantization on NCS

- Analyze the impact on closed-loop stability and performance
- Given the number of quantization bits, does there exist a stabilizing controller?

Data rate limitations and quantization

Reference setup



- Controller embedded in the encoder
- Ideal communication channel: $\hat{u}_k^{\text{coded}} = u_k^{\text{coded}}, k \geq 0$
- Shannon's theorem: maximal transmission rate of the channel

$$R = B \log_2(1 + SNR)$$

  - B: channel bandwidth [Hz]
  - SNR: signal-to-noise ratio in linear scale
  - R: max transmission rate $[\frac{\text{bit}}{\text{s}}]$

Next: focus on networks where R is low, e.g. wireless links based on Bluetooth or IEEE 802.11(b)

Packet network, decoder and encoder

Header (H)	Payload (P)

- N_B: n° of bits in each packet
- Quantized input: $u_k^{\text{coded}} \in \mathcal{U} = \{\bar{u}_1, \dots, \bar{u}_N\}$. The set $\mathcal{U}$ of admissible input values is known both to the encoder and the decoder. If $u_k^{\text{coded}} = \bar{u}_l$, the binary coding of the index l is transmitted and $\bar{u}_l$ is produced by the decoder
- For simplicity, no header $\rightarrow$ all bits used for representing the index l of $\bar{u}_l$
- Scalar control variable $\rightarrow$ $N = 2^{N_B}$ values

Example: $N_B = 1 \Rightarrow$ the index l can only take the values 0 and 1. The decoded signal $\hat{u}(t)$ can only take the values $u_{\min}$ and $u_{\max}$

Problem statement

Fundamental trade-off in network design

R is given. Choose N_B and the uniform sampling interval T. Ideally,

- T as small as possible $\rightarrow$ more reactive control
- N_B as large as possible $\rightarrow$ finer quantization

However, the time needed for transmitting N_B bits packet is $T_{\text{packet}} = N_B/R$ (we assume for simplicity zero link latency). The packet must arrive at the destination before the sample interval expires, i.e. $N_B/R \leq T$.

Fundamental inequality: $\dfrac{N_B}{T} \leq R$

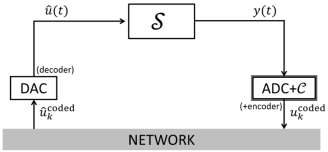
- Finer quantization $\Rightarrow$ higher simpling period
- Smaller sampling period $\Rightarrow$ coarser quantization

Key problems

- minimum value $R_{\min}$ of N_B/T that allows one to "stabilize" the NCS?
- if $N_B/T > R_{\min}$, how to design a quantized stabilizing controller?

Example: first-order systems

- $\mathcal{S}$: $\dot{x}(t) = ax(t) + bu(t)$
- Assumptions: $N_B = 1\frac{bit}{packet}$, T=1s, R=1.



- Sample-and-hold actuators $\rightarrow$ discrete-time system $\mathcal{S}^D$
$$x_{k+1} = fx_k + gu_k,$$
where $f = e^{aT}, g = -\frac{b}{a}(1 - e^{aT})$, if $a \neq 0$.
- Set $b = 1$ and study the control law
$$u_k = \begin{cases} -1 & \text{if } x_k \geq 0 \\ 1 & \text{if } x_k < 0 \end{cases}$$
corresponding to the set of admissible control values $\mathcal{U} = \{-1, 1\}$.

Closed loop dynamics

$$x_{k+1} = \begin{cases} fx_k - g & \text{if } x_k \geq 0 \\ fx_k + g & \text{if } x_k < 0 \end{cases}$$

Case I: $a = -1$

- $\mathcal{S}$: $\dot{x} = -x + u$ is AS
- $\mathcal{S}^D$: $x_{k+1} = 0.37x_k + 0.63u_k \Rightarrow$ AS (with no quantization)
- Closed-loop system:

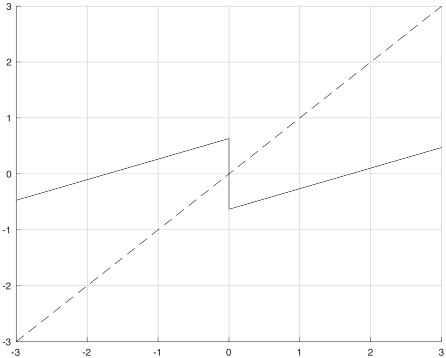
$$x_{k+1} = \begin{cases} 0.37x_k - 0.63 & \text{if } x_k \geq 0 \\ 0.37x_k + 0.63 & \text{if } x_k < 0 \end{cases}$$

Interlude - Graphical method for computing the closed-loop system

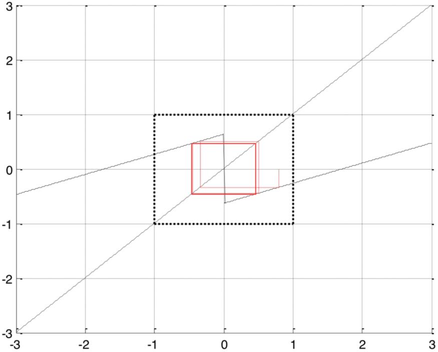
System $x_{k+1} = \tilde{f}(x_k)$, $\tilde{f}(\cdot)$ shown in the figure.
Input: x_0
For $k = 0, 1, 2, \ldots$

- From (x_k, x_k) (point on the diagonal) move vertically to cross $\tilde{f}(x_k)$ (so generating (x_k, x_{k+1})), then move horizontally to cross the diagonal (so generating (x_{k+1}, x_{k+1}))

The projection of points (x_k, x_k) on the horizontal axis gives $x_0, x_1, x_2, \ldots$



Case I: $a = -1$



Conclusions

For all x_0, the closed-loop state trajectory converge to $[-0.63, 0.63]$ and are eventually confined there.

- $[-0.63, 0.63]$ is positively invariant[a]
- states do not converge to zero

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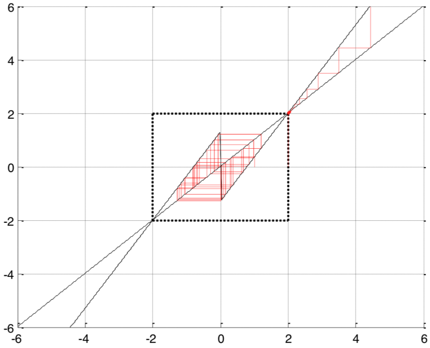
[a] A set $\mathcal{I}$ is positive invariant for $x^+ = f(x)$ if $x \in \mathcal{I} \rightarrow x^+ \in \mathcal{I}$.

Case II: $a = 0.5$

- $\mathcal{S} : \dot{x} = 0.5x + u \Rightarrow$ unstable
- $\mathcal{S}^D$: $x_{k+1} = 1.65x_k + 1.3u_k \Rightarrow$ unstable (without quantization)
- Closed-loop system:

$$x_{k+1} = \begin{cases} 1.65x_k - 1.3 & \text{if } x_k \geq 0 \\ 1.65x_k + 1.3 & \text{if } x_k < 0 \end{cases}$$

Case II: $a = 0.5$



Conclusions

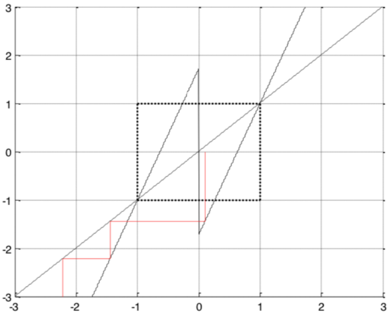
- if $x_0 \in (-2, 2)$, the trajectories tend to enter the set $[-1.3, 1.3]$, and are eventually confined there. Note that, however, the trajectories do not converge to zero!
- if $|x_0| = 2$, $x_k = x_0$ all $k \geq 0$. Indeed ± 2 are equilibria.
- if $|x_0| > 2$, $x_k \to \infty$ as $k \to +\infty$.

Case III: $a = 1$

- $\mathcal{S}$: $\dot{x} = x + u \Rightarrow$ unstable
- $\mathcal{S}^D$: $x_{k+1} = 2.7x_k + 1.7u_k \Rightarrow$ unstable (without quantization)
- Closed-loop system:

$$x_{k+1} = \begin{cases} 2.7x_k - 1.7 & \text{if } x_k \geq 0 \\ 2.7x_k + 1.7 & \text{if } x_k < 0 \end{cases}$$

Case III: $a = 1$



Conclusion

- For *almost all* x_0, one has $|x_k| \to +\infty$ as $k \to \infty$ (unbounded behavior)
- "Almost all" means for all x_0 which are neither the equilibria ± 1 nor the initial states from which an equilibrium can be reached in $N \in \mathbb{N}$ steps. There are at most 2^N states[a] with the latter property: 2^{N-1} of them end in $+1$ and the remaining 2^{N-1} end in -1.

[a]Exactly 2^N states, if they are all different.

Conclusions from the examples

For the proposed control law

1. $a < 0$ (AS CT system) $\rightarrow$ closed-loop states are bounded ("stable behavior")
2. $a > 0$ (unstable CT system) The closed-loop system has
   - bounded states trajectories if a is small enough
   - unbounded state trajectories if a is big enough

Next

- Clarify the "stability" properties that an NCS subject to quantization can enjoy
- Clarify what kind of open-loop instabilities can be compensated (e.g. how "small" $a > 0$ should be in (ii))

Boundability

Definition:

Consider the LTI system

$$\mathcal{S}: \quad \dot{x}(t) = Ax(t) + Bu(t), \qquad x(t) \in \mathbb{R}^n, u(t) \in \mathbb{R}^m$$

together with an admissible control set $\mathcal{U} \subseteq \mathbb{R}^m$. The system is boundable if there exists a bounded set $\mathcal{I} \subset \mathbb{R}^n$ and an open set[a] $\mathcal{M} \subset \mathcal{I}$ such that, for all $x(0) \in \mathcal{M}$ there is a discrete time control sequence $U = \{u_0, u_1, ...\}$, $u_i \in \mathcal{U}, i \geq 0$ guaranteeing that the closed-loop continuous-time state obtained by applying U in a sample-and-hold fashion lies in $\mathcal{I}$ at all times.

[a] A countable set of points $\{x_\ell\}_{\ell=1}^{+\infty}$ (such as the set of initial conditions providing a bounded state trajectory in the example with $a = 1$) is never an open set.

Boundability

Remarks on boundability

- Independent of the specific control law
- Case of interest for quantization: $\mathcal{U}$ is a finite set
- $\mathcal{S}$ AS $\Rightarrow \mathcal{S}$ boundable

  Proof:

  - choose $u_k = \bar{u} \in \mathcal{U} \Rightarrow x(k)$ converges to the equilibrium $\bar{x} = (I - A)^{-1} B \bar{u}$
  - it is also possible to show that an arbitrarily large positive invariant set centered at $\bar{x}$ exists (sublevel sets of a Lyapunov function certifying AS, if $\bar{x} = 0$)
- The only interesting case is when $\mathcal{S}$ is not AS (see the previous examples with $a > 0$)

The case of first-order systems

Assumptions:

- $\mathcal{S} : \dot{x} = ax + bu, b > 0$ (all results can be easily generalized to $b < 0$)
- Sample-and-hold scheme $u(t) = u_k$ for $t \in [kT, (k+1)T), k \geq 0$.
- $\mathcal{U}$ contains $N = 2^{N_B}$ elements in $[u_{min}, u_{max}]$
- The control law is piecewise-constant state feedback

$$u_k = s(x_k), \quad s(\cdot) : \text{selection function}$$

Recall: discrete-time open-loop dynamics

$$\mathcal{S}^D : x_{k+1} = f x_k + g u_k, \quad f = e^{aT}, g = b \int_0^T e^{a\tau} d\tau$$

Theorem (boundability)

Under the previous assumption, there exists a set $\mathcal{U}$ such that $\mathcal{S}$ is boundable *if and only if*

$$\frac{N_B}{T} \geq a \log_2 e = a \cdot 1.4427 \tag{1}$$

Moreover, there is a function $s(\cdot)$ steers the state to a bounded set.

Comments on the theorem

- (1) is called rate inequality
- Previous example on scalar systems ($\frac{N_B}{T} = 1$)
  - Case I: $a < 0$, $\frac{N_B}{T} \geq a \log_2 e \rightarrow$ Boundable
  - Case II: $a = 0.5$, $\frac{N_B}{T} \geq a \log_2 e \approx 0.72 \rightarrow$ Boundable
  - Case III: $a = 1 \rightarrow$, $\frac{N_B}{T} < a \log_2 e \approx 1.44 \rightarrow$ Not boundable
- A static state-feedback is enough for boundability

Construction of $s(\cdot)$ when operating at the rate limit

- Assume that $\frac{N_B}{T} = a \log_2 e$ and u_{min}, u_{max} are fixed
- $N = 2^{N_B}$, define $U = \{\bar{u}_0, \dots, \bar{u}_{N-1}\}$ where

$$\bar{u}_i = u_{max} + \frac{i}{N-1}(u_{min} - u_{max})$$

(equally spaced values in $[u_{min}, u_{max}]$)

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$$s(x) = \begin{cases} \bar{u}_0 = u_{max} & \text{if } x \le \bar{x}_1, \\ \bar{u}_i & \text{if } \bar{x}_i < x \le \bar{x}_{i+1}, i = 1, \dots, N-1 \\ \bar{u}_{N-1} = u_{min} & \text{if } x > \bar{x}_{N-1} \end{cases}$$

where for $i = 0, \dots N$

$$\bar{x}_i = -\frac{b}{a} u_{\max} + \frac{b}{a} \left(u_{\max} - u_{\min} \right) \frac{i}{N}$$

($\bar{x}_i$ are equally spaced values in $[-\frac{b}{a} u_{max}, -\frac{b}{a} u_{min}]$)

One has that (proof not shown):

- $\mathcal{M} = [\bar{x}_0, \bar{x}_N]$ is a positively invariant set for the closed-loop system
- if $x_0 \notin \mathcal{M}$, then $|x_k| \to +\infty$ as $k \to \infty$

Remarks

- $s(\cdot)$ is the only control law guaranteeing boundability if $\frac{N_B}{T} = a \log_2 e$.
- $s(\cdot)$ guarantees boundability even if $\frac{N_B}{T} > a \log_2 e$. In this case, however, more performing control laws can exist (e.g., using non equally-spaced quantization levels)

Higher-order systems

Consider the LTI system

$$\dot{x}(t) = Ax(t) + Bu(t), x(t) \in \mathbb{R}^n, u(t) \in \mathbb{R}$$

with scalar input and assume (A, B) is controllable[1]
The theorem about boundability holds provided that the data rate inequality (1) is replaced by

$$R \geq \frac{N_B}{T} \geq (Re(\lambda_1(A)) + \ldots + Re(\lambda_K(A))) \log_2 e$$

where $\lambda_1(A), \ldots, \lambda_K(A)$ are the eigenvalues of A with positive real parts

[1] i.e., the reachability matrix $\mathcal{M}_R = [B, AB, \ldots, A^{n-1}B]$ is full rank

Take-home messages

- Quantization effects due to limited bandwidth can substantially impact on the behavior of NCS
  - Quantization introduces a nonlinearity
  - Convergence to the origin might be impossible. Use boundability instead.
- The design of controllers for guaranteeing boundability can be non-trivial