Lecture 2

Linear Matrix Inequalities. Control Networks

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Recap from last lecture

LTV DT model

$$x(k+1) = A(k)x(k) + B(k)u(k)$$

Stability of the system = stability of the equilibrium $(\bar{x}, \bar{u}) = (0, 0)$

Lyapunov theorems with candidate Lyapunov function $V(x) = x^T P x$

- For the LTI system $x(k+1) = Ax(k)$

$$\text{AS/ES} \iff \exists P = P^T > 0 \text{ verifying } A^T P A - P < 0 \qquad (1)$$

- For the DT linear switched system $x(k+1) = A_{\sigma(k)}x(k)$, $\sigma(k) \in \mathcal{I} = \{1, \dots, M\}$

$$\exists P = P^T > 0 \text{ verifying } A_i^T P A_i - P < 0, \quad \forall i \in \mathcal{I} \Rightarrow \text{ES} \qquad (2)$$

Problem

How to check the existence of P verifying the inequalities in (1) and (2)?

Outline

- Introduction to Linear Matrix Inequalities (LMIs)
- Control networks: basics and performance analysis
  - Physical properties of communication links
  - Delays in control networks
  - Packet collisions and MAC protocol
  - Wireless control networks

Very short introduction to linear matrix inequalities

Definition

A Linear Matrix Inequality (LMI) is an inequality $F(X) > 0$ where

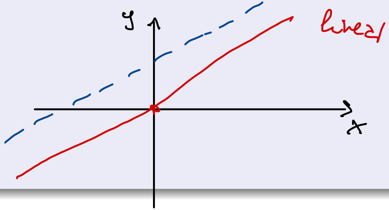
$$F : V \to S^n, \quad S^n = \text{set of symmetric } n \times n \text{ matrices}$$

is an affine function and V is a finite dimensional vector space

Remarks

- $F(X) > 0$ means the matrix $F(X)$ is positive-definite

affine = linear + constant



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- $F(X)$ is an affine function if $F(X) = F_0 + T(X)$ where $F_0 \in S^n$ and $T(X) : V \to S^n$ is a linear function

Very short introduction to linear matrix inequalities

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- Let $e_1 \ldots, e_m$ be a basis for V and $X = \sum_{i=1}^m \theta_i e_i$, $\theta_i \in \mathbb{R}$, $i = 1, \ldots, m$. Then, $T(X) = \sum_{i=1}^m \theta_i T(e_i)$, i.e. T is a linear combination of symmetric matrices

Very short introduction to linear matrix inequalities

LMI and control theory

Case of interest for control theory: $F : \mathbb{R}^{m_1 \times m_2} \to S^n$, i.e. the variable X of $F(X)$ is a matrix

Very short introduction to linear matrix inequalities

LMI and control theory

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Example - Stability test for LTI systems

The discrete-time system $x(k+1) = Ax(k)$, $x(k) \in \mathbb{R}^n$ is AS iff $\exists P \in S^n$ such that

$$P > 0 \tag{3}$$

$$A^T PA - P < 0 \tag{4}$$

(3) and (4) are matrix inequalities. Are they LMI ? Yes because

- (3) is $F_1(P) > 0$ with $F_1(P) = P$, which is affine in the unknown P. Moreover $F_1(P) = F_1(P)^T$
- (4) is $F_2(P) > 0$ with $F_2(P) = -A^T PA + P$, which is affine in P. Moreover it is easy to show that $F_2(P) = F_2(P)^T$.

$F_2(P)^T = -A^T P^T A + P^T = -A^T PA + P$ $F_2(P)$

Very short introduction to linear matrix inequalities

LMI systems

Proposition. The system of LMIs

$$\begin{cases} F_1(X) > 0 \\ \vdots \\ F_p(X) > 0 \end{cases}$$

$A_1 > 0$
$A_2 > 0$
$\Updownarrow$
$\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} > 0$

is equivalent to the single LMI $\mathrm{diag}(F_1(X), \ldots, F_p(X)) > 0$

Very short introduction to linear matrix inequalities

LMI systems

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Example - (ctd.)

The system $x(k+1) = Ax(k)$, $x(k) \in \mathbb{R}^n$ is asymptotically stable iff $\exists P \in S^n$ such that

$$\begin{bmatrix} P & 0 \\ 0 & -A^T PA + P \end{bmatrix} > 0$$

Very short introduction to linear matrix inequalities

LMI optimization problem

$$\min_X c(X)$$

subject to

$$\begin{cases} F_1(X) > 0 \\ \vdots \\ F_p(X) > 0 \end{cases}$$

where $c(X)$ is a linear function and $F_i(X) > 0$ are LMIs

LMI feasibility problem

Check if there is X verifying the constraints

$$\begin{cases} F_1(X) > 0 \\ \vdots \\ F_p(X) > 0 \end{cases}$$

Very short introduction to linear matrix inequalities

Remarks

- LMI feasibility and optimization problems are *convex programming problems* for which there are efficient (i.e. polynomial-time) algorithms. Free software in MatLab:
  - LMI control toolbox
  - SDPT3 toolbox
  - SeDuMi toolbox
  - ... and many others

Very short introduction to linear matrix inequalities

Remarks

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  - LMI control toolbox
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  - ... and many others
- Tons of interesting problems in control and engineering can be cast into LMIs. See, e.g. the book
  - Boyd, S. and Vandenberghe, L., V. *Convex optimization*, Cambridge University Press, 2004.
- In this course: LMIs for analyzing stability of NCSs

Example: From LMI to MatLab Code

- Quadratic Lyapunov Function: LMI's

$$\begin{cases} A^T PA - P < -Q \\ P > 0 \end{cases}$$

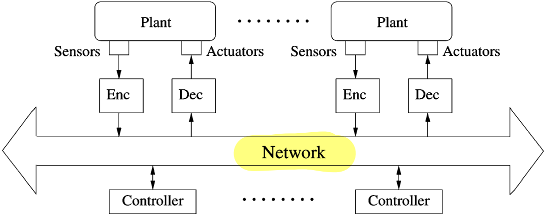
- MatLab + Yalmip code

```
A = 0.1*[-1 2 0;-3 -4 1;0 0 -2];
P = sdpvar(3,3); %Unknown 3x3 symmetric matrix
Q = 1/100 * eye(3,3);
L1 = [A'*P*A - P + Q < 0]; %Constr. 1
L2 = [P > 0]; %Constr. 2
L = L1 + L2; %Combine all constraints
solvesdp(L); %Solving for P (matlab workspace)
P = double(P); %Converts to standard format
```

More in the exercise session !

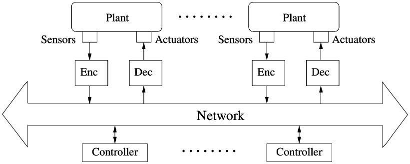
Control Networks: Basics and Performance Analysis

Networked Control System (NCS)



- Today we focus on the communication network
  - Goals: understand how it works and sources of delays and packet drop
  - Disclaimer: simplified description!
- NCSs use *control networks*. Why are they needed ?

Networked Control System (NCS)



Control networks vs Internet

Control networks

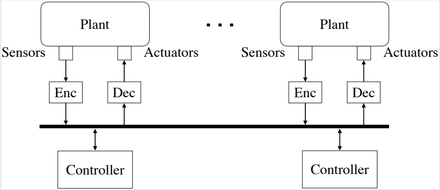
- have simpler topologies (no need of sophisticated routing)
- devices simpler than computers (e.g. a microcontroller does not run several applications in parallel requiring the network)
- shuttle small but frequent packets
- aim at meeting time-critical requirement $\Rightarrow$ support real-time or time-critical applications !

Ideal goal of control nets: transmit a message within a bounded and small time-delay !

Networked Control System

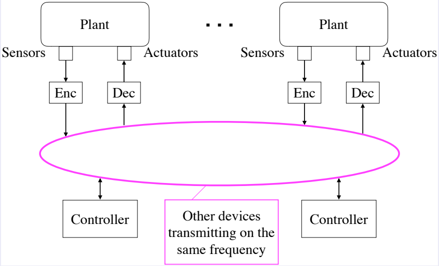
Reference topologies

Bus network



The most frequent topology for a control network

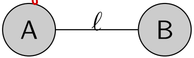
Wireless network (broadcast)

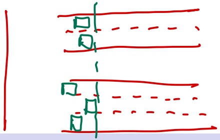


- Shared medium: how to access it minimizing conflicts ?
- In the sequel: focus on a single link

Nodes and links

$\frac{1}{\tilde{B}}$ = time it takes for a single bit to access the network



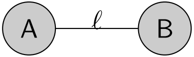


Physical properties of the link ℓ

$\tilde{B}$ does not depend on the length ℓ

1. ℓ = low-pass filter with bandwidth B [Hz]
2. Signal-to-noise (S/N) ratio
   $\rightarrow$ **Shannon's theorem:** every link has a maximal transmission rate
   $\tilde{B}$ = max n° of bits/sec$= B\log_2(1 + \mathrm{S/N})$
   $\tilde{B}$ measured in bits per second (bps). Also called "Bandwidth" in computer science
   $\rightarrow$ Remark: if S/N is not constant, $\tilde{B}$ changes as well!

Nodes and links



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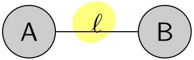
Example - Telephone line (ADSL)

Link bandwidth: 1 MHz, S/N: 10000$\Rightarrow$ max n° of bps
$= 10^6\log_2(1+10000) \simeq$ 13 Mbps

Nodes and links



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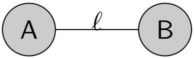
Link bandwidth: 1 MHz, S/N: 10000$\Rightarrow$ max n° of bps
$= 10^6\log_2(1+10000) \simeq$ 13 Mbps

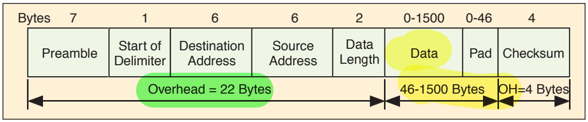
3. Latency (delay): propagation time [s] for 1 bit to travel along the link
   $\Rightarrow$ usually proportional to the length of ℓ

Packet networks



- Data is transmitted in atomic units called *packets*[1]

Ethernet packet



[Lian et al., '01]

- Roughly, a packet is composed of a header and a data field
- Packets can have different sizes, depending on the data field
- Transmitting 1 bit of data or several bytes always costs 1 packet

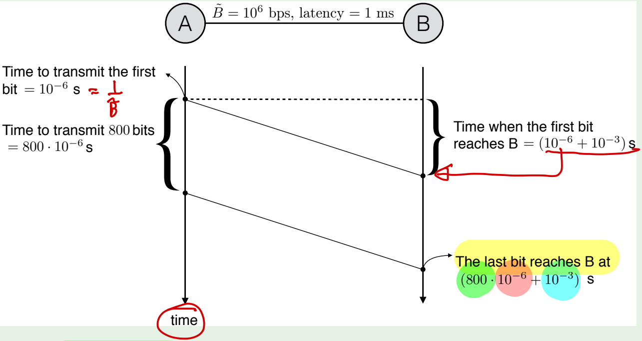
[1] At the link level, packets are more correctly called "frames"

Delays in Control Networks

Packet delay

The whole packet must be transmitted $\Rightarrow$ additional delay source, on top of latency

Sending a 800 bit packet from A to B

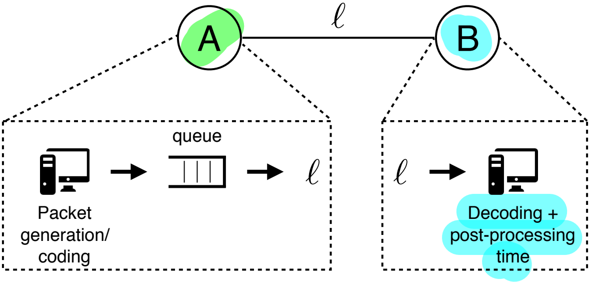


$$\text{Packet delay} = \frac{\text{Packet size}}{\tilde{B}} + \text{link latency}$$

Deterministic delay component if the S/N is constant (not true for wireless...)

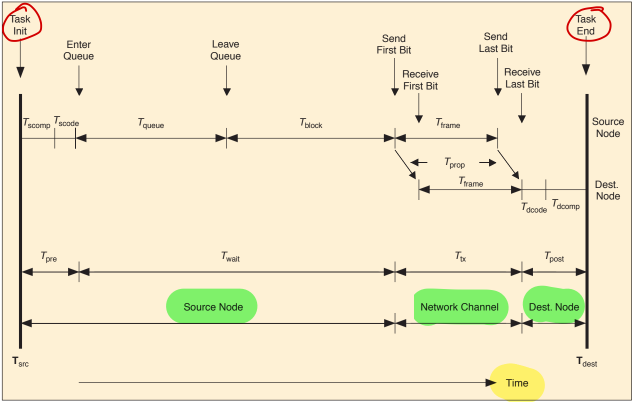
Other sources of delays

- Source nodes are equipped with queues needed for resolving conflicts
  - Delay due to queuing time = time a message waits in the queue while previous messages in the queue are sent
    - ★ Depends on the network load and protocol (see next) → stochastic delay component
- Destination nodes need to decode and post-process packets before the data can be used → additional delay



Other sources of delays

Summarizing



[Lian et al., '01]

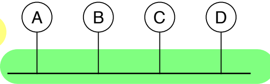
Several sources, three main categories (source node, network channel, destination node)

2h22

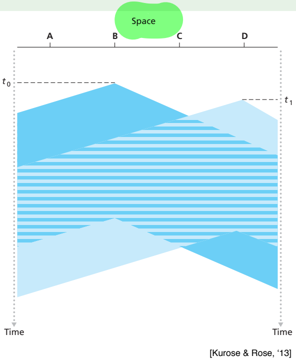
Packet collisions and the MAC protocol

Packet collision

- Premise: nodes can sense if the bus is free all the time
- If they follow the rule of transmitting only when the bus is free (Carrier Sense Multiple Access (CSMA) rule), why collisions happen ?



Space-time diagram: B and D transmit



- At time t_0, B senses the bus is free and starts transmitting
- At time t_1, D senses the bus is free and starts transmitting
  $\rightarrow$ Collisions !
- The longer the bus, the higher the probability of collision

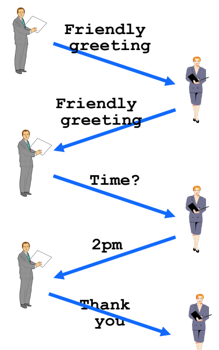
Collision management

- Nodes can detect collision (sensed $\neq$ transmitted)
- Retransmit the packet ? Who retransmits ?

  $\Rightarrow$ Need of a Medium Access Control (MAC) protocol !

MAC protocol

What is a protocol?



- Agreement between different devices about network access
- The MAC protocol influences a lot delays and packet losses (see next)

  $\Rightarrow$ it is a "non physical" source of packet loss and delays

Next: compare 3 popular types of control networks

Bus topology: 3 different MAC protocols

- Ethernet with "Carrier Sense Multiple Access with Collision Detection (CSMA/CD)"
- Token-passing (e.g. ControlNet)
- Controller Area Network (CAN) (e.g. DeviceNet)

Bonus: wireless control networks

Ethernet CSMA/CD (simplified description)

- When a node wants to transmit, it listens to the network (busy = wait)
- Two nodes transmit at the same time → messages collide and get corrupted
  ⇒...but nodes listen while transmitting and detect collision
- Collision detected: the transmission node stops, waits a random time and retransmits
  ⇒ after 16 collisions, the node drops the packet and tells it to the microprocessor (the "packet generator")

Pros

Simple MAC protocol → almost no delay at low network loads

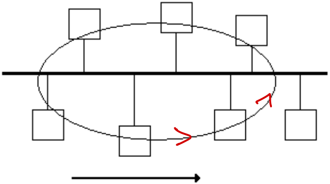
Cons

Nondeterministic protocol. At high network loads delays may be unbounded

Token-passing bus (e.g. ControlNet)

Nodes arranged logically in a ring

- The node with the token transmits until
  - it has no more data or
  - the max time for holding a token is reached
- The token is passed to the successor



Pros

- Data frames never collide
- Transmission delay bounded by the token rotation time !
- Easy to add nodes
- Excellent throughput at high network loads

Cons

- Limited n° of nodes (1, . . . , 99) [needed for implementing implicit token passing through addresses] ⇒ each node must know which is the next one (unique MAC ID)
- Less efficient then CSMA/CD at low traffic, because token-passing introduces overhead

CAN bus with CSMA/Arbitration Message Priority (e.g. DeviceNet)

- Each message has a priority, used to arbitrate access to the bus in case of simultaneous transmissions
- A node that wants to transmit waits until the bus is free. Then:
  - starts sending the message identifier (11 bits) bit-by-bit (a logic 0 is dominant on a logic 1)
  - All nodes have synchronized clocks for detecting the start of a bit-period

    $\Downarrow$

    In this phase, arbitration is performed and as soon as a node receives a bit different from the one it sent, it stops sending his message $\Rightarrow$ An ongoing transmission is NEVER corrupted !

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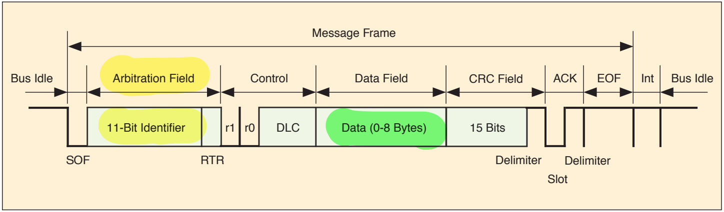
The destination/source unit might not even be specified, but the message identifier is unique in the network. All units listen and discard messages they are not interested in. This is called *multicast*.

CAN bus with CSMA/Arbitration Message Priority (e.g. DeviceNet)

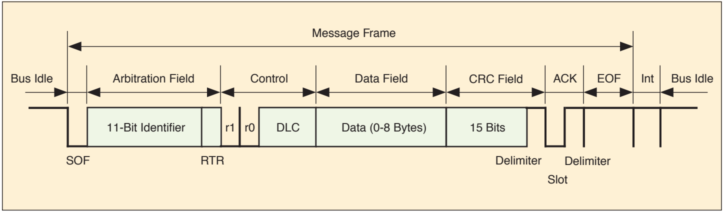


[Lian et al., '01]

Pros

- Deterministic protocol, optimized for short messages
- Transmission of high-priority messages is guaranteed with a given maximal delay
- An ongoing transmission is never corrupted

CAN bus with CSMA/Arbitration Message Priority (e.g. DeviceNet)



[Lian et al., '01]

Cons

- Keeping precise clock synchronization requires
  - slow transmission rate (max 500 kb/s)
  - short cable length
- Variable delay for low-priority messages (that must be promoted to high-priority for increasing chances to be transmitted)

Typical parameters of control networks

(token-passing) (CAN)

Bandwidth

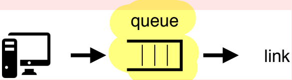
Table 1. Typical system parameters of control networks.

	Ethernet	ControlNet	DeviceNet
Data rate[a] (Mb/s)	10	5	0.5
Max. length (m)	2500	1000	100
Max. data size (bytes)	1500	504	8
Min. message size[b] (byte)	72[c]	7	47/8[d]
Max. number of nodes	>1000	99	64
Typical *Tx* speed (m/s)	Coaxial cable: 2×10^8		

a: typical data rate;
b: zero data size;
c: including the preamble and start of delimiter fields;
d: DeviceNet overhead is 47 bits.

General remark

Retransmission, clock synchronization and token passing require to implement a queue at the source node, in order to decouple transmission from the functioning of the microprocessor



Case study on network-induced delays: 10 nodes network

- Each node uses a sampling time (aka "message period") of 5000 μs (chosen so that network is not saturated)

5ms

- Each node sends 8 bytes in every period. Three release policies:
  1. "Zero": all nodes start transmitting at the beginning of the period
  2. "Random": the beginning of transmission is chosen randomly within each period
  3. "Scheduled": pre-specified beginning-of-transmission time for each node within each period

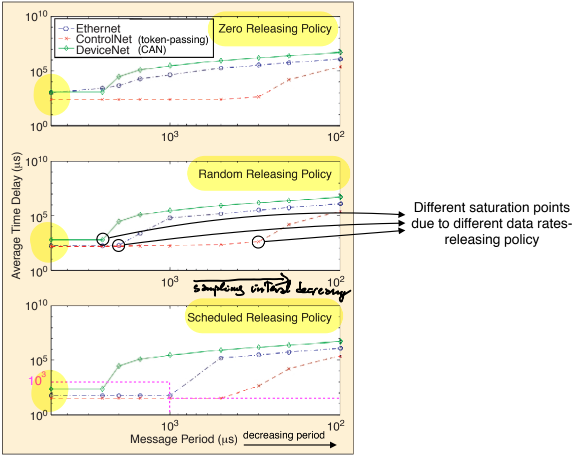
Table 2. Simulation result of three releasing policies with message period of 5000 µs (ten-node case).

Releasing Policies	Zero	Random	Scheduled
Average time delay (µs)			
Ethernet	1081	172	58
ControlNet (token-passing)	241	151	32
DeviceNet (CAN)	1221	620	222

Main message

Delays also depend on how packets are released (on top of the sources of delays previously analyzed)

Average delay as a function of the sampling time



[Lian et al., '01]

- New experiments where the sampling time is varying (5000 μs is the origin of the horizontal axis)
- Total delays from the packet generation to the packet post-processing
- Packets arrived after the end of the sampling interval are discarded, all networks suffer from packet drops (time-varying and random, as the delays)

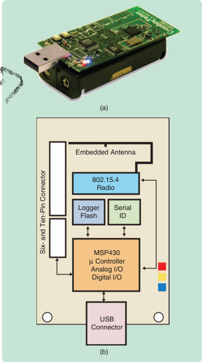
Main message

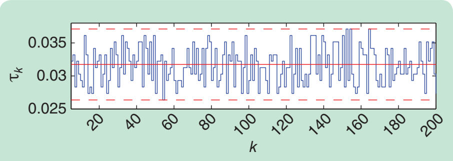
Delays also depend on the sampling time (on top of the sources of delays previously analyzed)

Wireless control networks

[Bauer et al., '14]

- Experiment: output-feedback control of an inverse pendulum on a cart
- Sensors transmit position and angle to the controller
- Telos B motes communicating in the 2.4 GHz band implement the wireless link from the sensors to the controller
- MAC protocol: Token-passing-like $\Rightarrow$ avoids packet losses if NO other device is using the 2.4 GHz band (e.g. Bluetooth, WiFi, etc.)





Main message

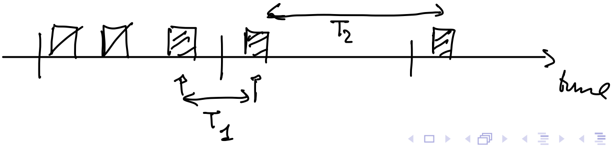
Delays also depend on other devices using the same band and vary in a stochastic fashion

Time-varying sampling intervals in control networks

Why sampling intervals experienced by the controller might be time-varying?

- Retransmission after conflict detection causes fluctuations around a nominal duration of the sampling time
  - Packet dropouts are caused only by *multiple consecutive* conflicts
- Some MAC protocols can modify the sampling intervals for reducing the network load

sensor with multiple sampling within an interval



Take-home messages

- Control networks aim at supporting real-time operations (small and frequent packets)
- Delays are induced by
  - the physical layer
  - the MAC protocol

  ...and are time-varying, often stochastic
- Packet dropouts due to
  - collisions + no retransmission of old packets
- Sampling intervals can be time-varying

1h 03