Networked Control Systems (ME-427)- Exercise session 14

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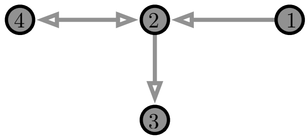
1. **Laplacian average consensus in directed networks.** [Textbook E7.4] Consider the directed network in figure below with arbitrary positive weights and its associated Laplacian flow $\dot{x}(t) = -Lx(t)$.

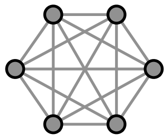


   (a) Can the network reach consensus, that is, as $t \to \infty$ does $x(t)$ converge to a limiting point in $\text{span}\{\mathbb{1}_n\}$?

   (b) Does $x(t)$ achieve average consensus, that is, $\lim_{t\to\infty} x(t) = \text{average}(x(0))\mathbb{1}_n$?

   (c) Will your answer to point (b) change if you smartly add one directed edge and adapt the weights?

2. **The adjacency and Laplacian matrices for the complete graph.** [Textbook E6.2] For any number $n \in \mathbb{N}$, the *complete graph* with n nodes, denoted by $K(n)$, is the undirected and unweighted graph in which any two distinct nodes are connected. For example, see $K(6)$ in figure.



   Compute, for arbitrary n,

   (a) the adjacency matrix of $K(n)$ and its eigenvalues; and

   (b) the Laplacian matrix of $K(n)$ and its eigenvalues.

   Hint. Note that if $\lambda \in \text{Spec}(A)$ then $\lambda + \alpha \in \text{Spec}(A + \alpha I)$. Indeed, if $v \neq 0$ verifies $Av = \lambda v$, then ...

3. **Linear spring networks with loads.** [Textbook E6.15] Consider the two (connected) spring networks with n moving masses in figure. For the right network, assume one of the masses is connected with a single stationary object with a spring. Refer to the left spring network as *free* and to the right network as *grounded*. Let F_{load} be a load force applied to the n moving masses.

   For the left network, let $L_{\text{free,n}}$ be the $n \times n$ Laplacian matrix describing the free spring network among the n moving masses, as defined the lectures. Let L_{grounded} be the $n \times n$ *grounded Laplacian* of the n masses taking into account the elastic force of the stationary object.

   For the free spring network subject to $F_{\text{load}} = \begin{bmatrix} F_{\text{load},1}, & \dots, & F_{\text{load,n}} \end{bmatrix}$,

(a) Show that, for any $x \in \mathbb{R}^n$, the vector $v = L_{\text{free,n}} x$ is balanced, that is $\mathbb{1}_n^T v = 0$.

(b) Do equilibrium displacements exist for arbitrary loads ? If the load force F_{load} is balanced, that is $\mathbb{1}_n^T F_{\text{load}} = 0$, is the resulting equilibrium displacement unique?

Hint: Start by writing the system synamics as $M\ddot{x} = \ldots$ where $M = \text{diag}(M_1, \ldots, M_n)$ collects the value of the masses and $x = [x_1, \ldots, x_n]^T$ the position of the masses.

For the grounded spring network,

(c) derive an expression relating L_{grounded} to $L_{\text{free,n}}$,

(d) one can show that L_{grounded} is invertible. Using this fact, compute the equilibrium displacement for the grounded spring network for arbitrary and constant load forces.

4. **Euler discretization of the Laplacian.** [Textbook E7.6] Given a weighted digraph G with Laplacian matrix L and maximum out-degree $d_{\max} = \max\{d_{\text{out}}(1), \ldots, d_{\text{out}}(n)\}$. Show that:

(a) if $\epsilon < 1/d_{\max}$, then the matrix $I_n - \epsilon L$ is row-stochastic,

(b) if $\epsilon < 1/d_{\max}$ and G is weight-balanced, then the matrix $I_n - \epsilon L$ is doubly-stochastic, and

(c) if $\epsilon < 1/d_{\max}$ and G is strongly connected, then $I_n - \epsilon L$ is primitive (**Hint:** $I_n - \epsilon L$ can be interpreted as the adjacency matrix of a new weighted digraph ...)

Given these results, we highlight that $I_n - \epsilon L$ is the one-step Euler discretization (with sampling time $\epsilon > 0$) of the continuous-time Laplacian flow and is a discrete-time consensus algorithm.