Networked Control Systems (ME-427)- Exercise session 12

Prof. G. Ferrari Trecate

1. **A sample DeGroot panel.** [Textbook E5.1] A conversation between 5 panelists is modeled according to the DeGroot model by an averaging system $x^+ = Ax$, where
$$A = \begin{bmatrix} 0.15 & 0.15 & 0.1 & 0.2 & 0.4 \\ 0 & 0.55 & 0 & 0 & 0.45 \\ 0.3 & 0.05 & 0.05 & 0 & 0.6 \\ 0 & 0.4 & 0.1 & 0.5 & 0 \\ 0 & 0.3 & 0 & 0 & 0.7 \end{bmatrix}.$$
Assuming that the panel has sufficiently long deliberations, answer the following:
   (a) Draw the condensation of the associated digraph.
   (b) Do the panelists finally agree on a common decision?
   (c) In the event of agreement, does the initial opinion of any panelists get rejected? If so, which ones?
   (d) Assume the panelists' initial opinions are their self-appraisals (i.e., the self-weights $a_{11}, \ldots, a_{55}$) and compute the final opinion (Hint: use MatLab for computing relevant eigenvectors).

2. **A stubborn agent.** [Textbook E5.5] Pick $\alpha \in]0, 1[$, and consider the discrete-time averaging algorithm
$$\begin{aligned} x_1(k+1) &= x_1(k), \\ x_2(k+1) &= \alpha x_1(k) + (1-\alpha)x_2(k). \end{aligned}$$
Perform the following tasks:
   (a) compute the matrix A representing this algorithm and verify it is row-stochastic,
   (b) compute the eigenvalues and eigenvectors of A,
   (c) draw the directed graph G representing this algorithm and discuss its connectivity properties,
   (d) compute the condensation digraph of G,
   (e) compute the final value of this algorithm as a function of the initial values invoking the theorem on consensus with globally reachable nodes.

3. **The equal-neighbor row-stochastic matrix for weighted directed graphs.** [Textbook E5.3] Let G be a weighted digraph with n nodes, weighted adjacency matrix A and weighted out-degree matrix D_{out}. Define the *equal-neighbor matrix*
$$A_{\text{equal-neighbor}} = (I_n + D_{\text{out}})^{-1}(I_n + A).$$
Show that
   (a) $A_{\text{equal-neighbor}}$ is row-stochastic;
   (b) $A_{\text{equal-neighbor}}$ is primitive if and only if G is strongly connected; and
   (c) $A_{\text{equal-neighbor}}$ is doubly-stochastic if G is weight-balanced and the weighted degree is constant for all nodes (i.e., $D_{\text{out}} = D_{\text{in}} = dI_n$ for some $d \in \mathbb{R}_{>0}$).

Hint: First, for any $v \in \mathbb{R}^n$ with non-zero entries, it is easy to see $\text{diag}(v)^{-1}v = \mathbb{1}_n$, where $\text{diag}(v)$ is the diagonal matrix with the elements of v on the main diagonal. Note also that, by definition, $D_{\text{out}} + I_n = \text{diag}((A + I_n)\mathbb{1}_n)$ and $D_{\text{in}} + I_n = \text{diag}((A + I_n)^T \mathbb{1}_n)$.

4. **Reversible primitive row-stochastic matrices.** [Textbook E5.4] Let A be a primitive row-stochastic $n \times n$ matrix and w be its left dominant eigenvector (i.e. the left eigenvector associated with the dominant eigenvalue). The matrix A is *reversible* if

$$w_i A_{ij} = A_{ji} w_j, \quad \text{for all } i, j \in \{1, \dots, n\}, \tag{1}$$

or, equivalently

$$\operatorname{diag}(w) A = A^T \operatorname{diag}(w).$$

Prove the following statements:

(a) if A is reversible, then its associated digraph is undirected, that is, if (i, j) is an edge, then so is (j, i)

(b) if A is reversible, then $\operatorname{diag}(w)^{1/2} \cdot A \cdot \operatorname{diag}(w)^{-1/2}$ is symmetric and, hence, A has n real eigenvalues and n eigenvectors. Recall that, for $w = (w_1, \dots, w_n) > 0$, the following definitions hold: $\operatorname{diag}(w)^{1/2} = \operatorname{diag}(\sqrt{w_1}, ..., \sqrt{w_n})$ and $\operatorname{diag}(w)^{-1/2} = \operatorname{diag}(1/\sqrt{w_1}, ..., 1/\sqrt{w_n})$.

(c) If A is an equal-neighbor matrix for an unweighted undirected graph, then A is reversible. Using MatLab, verify this statement for the equal-neighbor matrix associated to the undirected graph $G = (V, E)$, $V = \{1, 2, 3, 4\}$, $E = \{(1, 2), (2, 3), (3, 4), (4, 2)\}$.