

Networked Control Systems (ME-427)- Exercise session 10

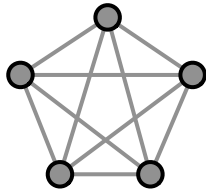
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1. Consider the linear averaging algorithm

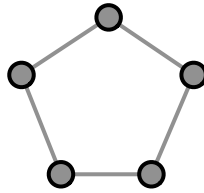
$$x_i^+ := \text{average}(x_i, \{x_j, \text{for all neighbor nodes } j\}). \quad (1)$$

Set $n = 5$, select the initial state equal to $(1, -1, 1, -1, 1)$, and use the following undirected un-weighted graphs (depicted in figure):

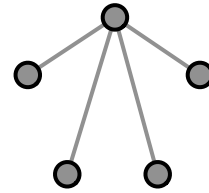
- (a) the complete graph,
- (b) the ring graph, and
- (c) the star graph with node 1 as center.
 - i. In MatLab, compute the consensus values using the results seen in the lecture.
 - ii. Verify the results through simulations (use the code developed in the previous exercise sessions).



(a) Complete graph



(b) Ring graph



(c) Star graph

2. **Discrete-time control of mobile robots.** Consider $n = 3$ robots moving on the line with positions $z_1, z_2, z_3 \in \mathbb{R}$. In order to gather at a common location (i.e., reach rendezvous), each robot heads for the centroid of its neighbors, that is,

$$\dot{z}_i = \frac{1}{n-1} \left(\sum_{j=1, j \neq i} z_j \right) - z_i.$$

Consider the Euler discretization of the above closed-loop dynamics with sampling rate $T > 0$:

$$z_i(k+1) = z_i(k) + T \left(\frac{1}{n-1} \left(\sum_{j=1, j \neq i} z_j(k) \right) - z_i(k) \right).$$

If $T \in [0, \frac{1}{2}]$, will rendezvous be guaranteed? If yes, to which position will the robots meet?

3. **The exponent of a primitive matrix.**

- (a) Let G be the digraph with nodes $\{1, \dots, 3\}$ and edges $(1, 2), (2, 1), (2, 3), (3, 1)$. Explain if and why G is strongly connected and aperiodic.
- (b) Recall a non-negative matrix A is primitive if there exists a number k such that $A^k > 0$; the smallest such number k is called the *exponent* of the primitive matrix A . Using G in point (a), show that k can be larger than the number of nodes.
- (c) Provide an intuitive motivation for the result in point (b), by reasoning on the meaning of elements $(A^k)_{ij}$.

4. Let $A \in \mathbb{R}^{n \times n}$ be a non-negative matrix. Prove that if A is irreducible, then $\tilde{A} = (A + I)$ is primitive.

5. **Leslie population model.** The Leslie model is used in population ecology to model the changes in a population of organisms over a period of time. In this model, the population is divided into n groups based on age classes; the indices i are ordered increasingly with the age, so that $i = 1$ is the class of the newborns. The variable $x_i(k)$, $i \in \{1, \dots, n\}$, denotes the number of individuals in the age class i at time k ; at every time step k the $x_i(k)$ individuals

- produce a number $\alpha_i x_i(k)$ of offsprings (i.e., individuals belonging to the first age class), where $\alpha_i \geq 0$ is a fecundity rate, and
- progress to the next age class with a survival rate $\beta_i \in [0, 1]$.

In the following, let $n = 3$. If $x(k)$ denotes the vector of individuals at time k , the Leslie population model reads

$$x(k+1) = Ax(k) = \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & 0 & 0 \\ 0 & \beta_2 & 0 \end{bmatrix} x(k) \quad (2)$$

where A is referred to as the Leslie matrix. Assume $\alpha_i > 0$ for all $i \in \{1, \dots, 3\}$ and $0 < \beta_i \leq 1$ for all $i \in \{1, \dots, 2\}$.

(a) Prove that matrix A is primitive.

(b) Let $p_i(k) = \frac{x_i(k)}{\sum_{i=1}^3 x_i(k)}$ denote the percentage of the total population in class i at time k . Call $p(k)$ the population distribution at time k . Compute by hand and in closed form $\lim_{k \rightarrow +\infty} p(k)$ as a function of the spectral radius $\rho(A)$ and the parameters (α_i, β_i) , $i \in \{1, \dots, 3\}$.