Networked Control Systems (ME-427)- Exercise session 6

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1. Consider the continuous-time system

$$\dot{x} = x + u$$

along with the network control system shown in Fig. 1, where the network is ideal. We aim to control the system using a quantized control law.

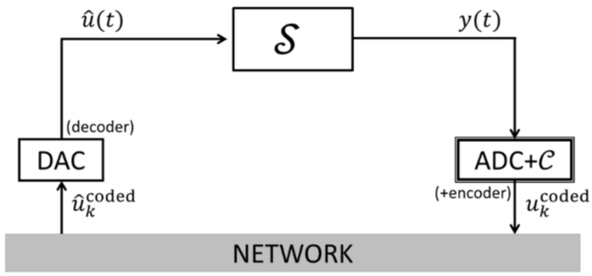


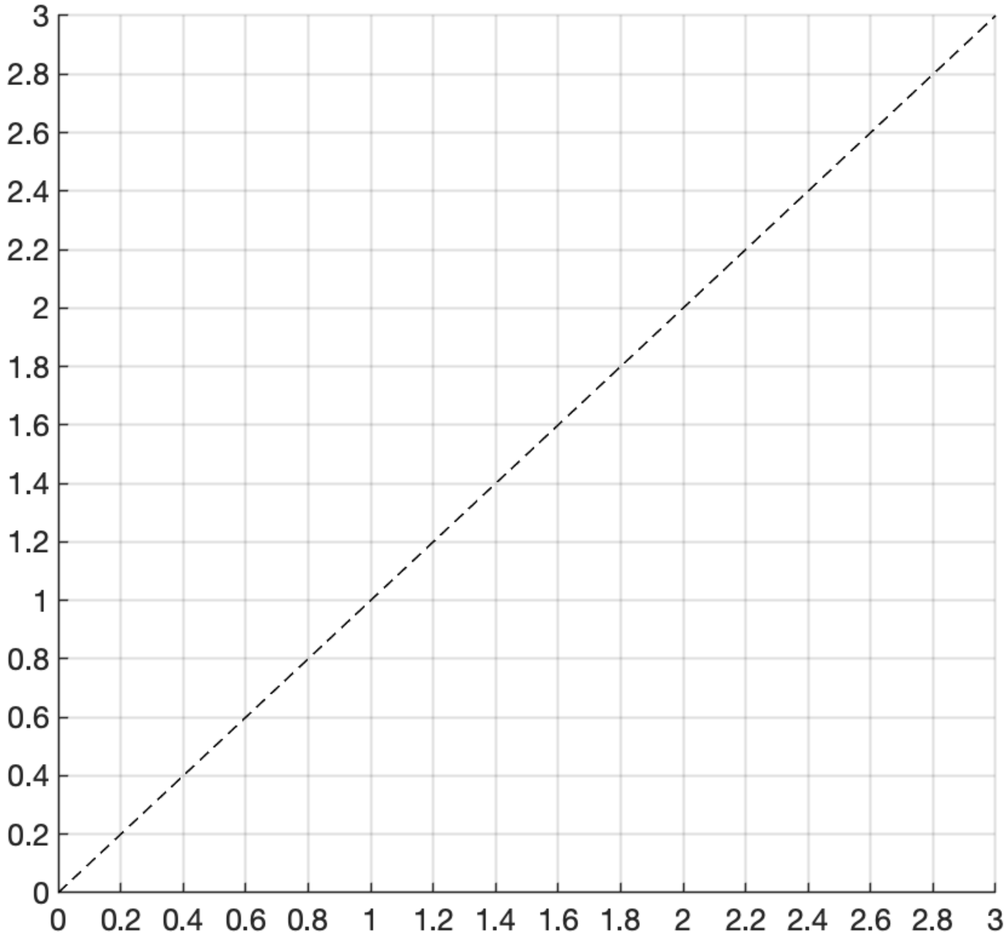
Figure 1: Networked control system

(a) According to the theory, what is the minimun rate $\frac{N_B}{T}$ required to make the system boundable?

(b) If we set the sampling time $T = \log_e(2.5) = 0.9163$ s, what is the minimum number of bits N_B required to be transmitted at each sampling instant in order to make the system boundable?

(c) Set $T = \log_e(2.5) = 0.9163$ s and assume that $N_B = 1$. Using the control law $u_k = -\text{sign}(x_k)$, show the boundaries of the state trajectories using the graphical method.

(d) Set $T = \log_e(2.5) = 0.9163$ s and assume that $N_B = 2$. Using the control law defined in case we operate at the rate limits (with $u_{\min} = -1, u_{\max} = 1$), study the boundaries of the state trajectories using the graphical method. Paper sheets for deriving relevant plots by hand are provided following this exercise. In case of boundability, define the corresponding positively invariant set.


3
2.8
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2.4
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1
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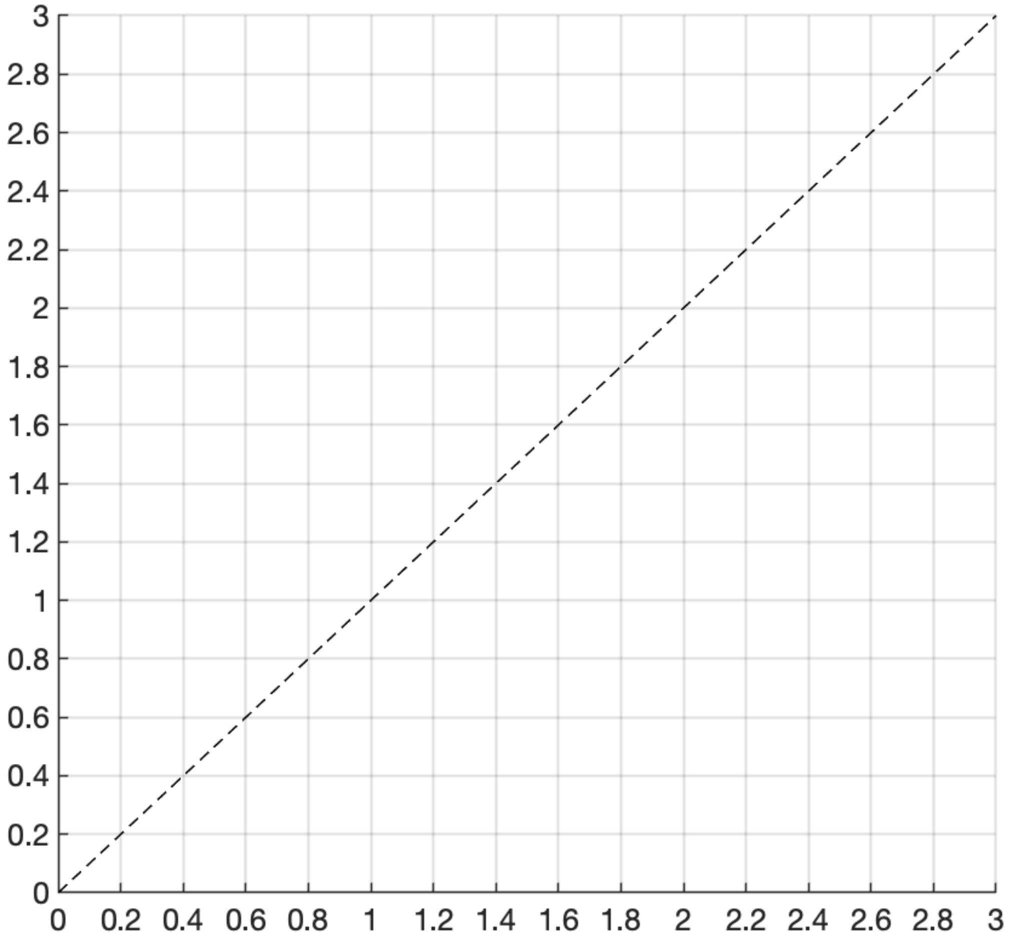
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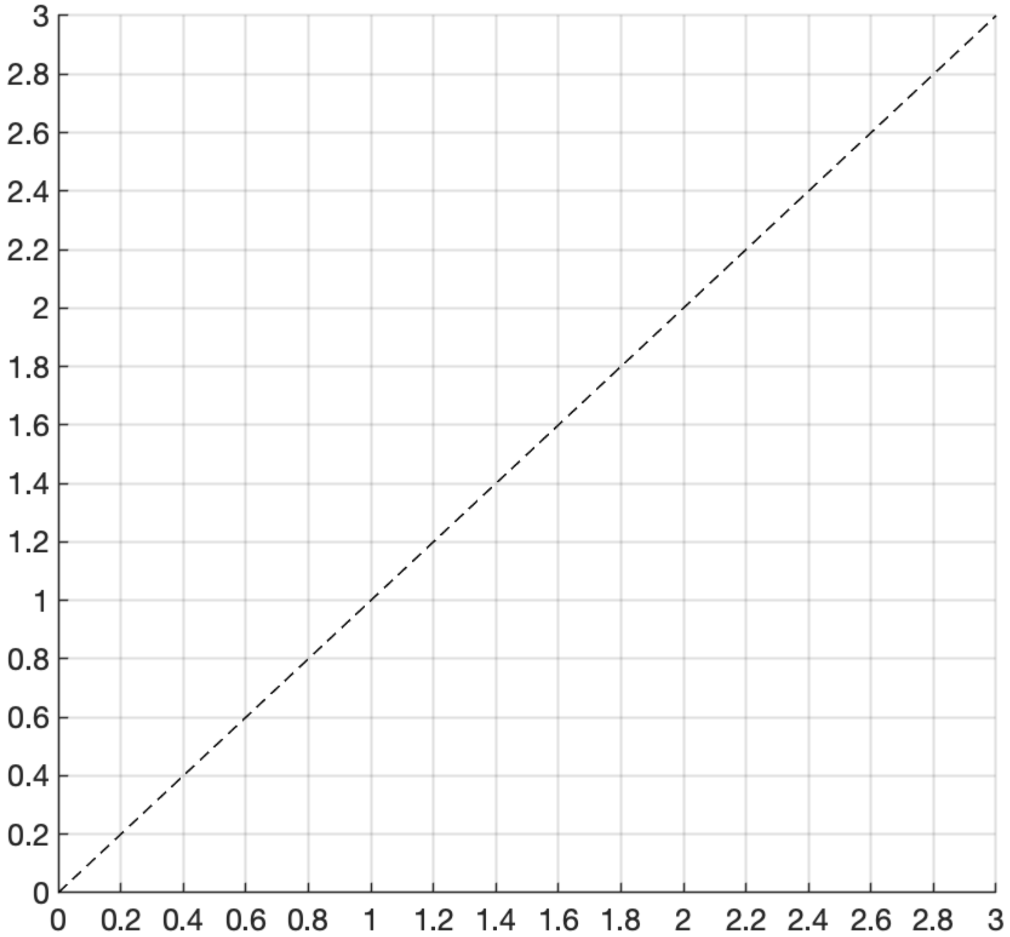
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3

Solution:

(a) In order to make the system boundable, the following inequality must be satisfied

$$\frac{N_B}{T} \geq a \log_2 e = 1.442 \text{ bit/s}.$$

(b) Setting $T = \log_e(2.5) = 0.9163$ s, the minimum number of bits N_B required to be transmitted at each sampling instant in order to make the system boundable is

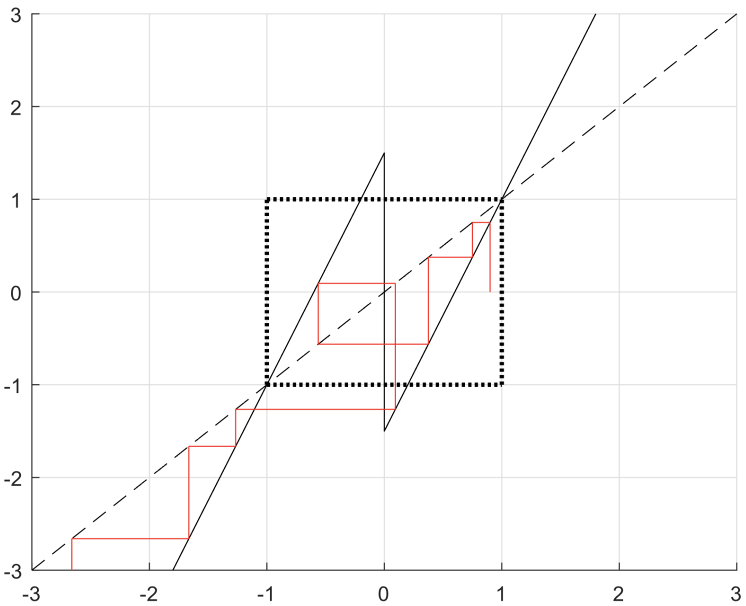
$$N_B \geq 1.442 \cdot 0.9163 = 1.32.$$

Therefore, being N_B an integer number, the minimum number of bits required to make the systen boundable is $N_B = 2$.

(c) Setting $T = \log_e(2.5) = 0.9163$ s, $N_B = 1$ and using the control law $u_k = -\text{sign}(x_k)$, the dynamics of the system under control is

$$\begin{cases} x_{k+1} = 2.5x_k + 1.5 & x_k \leq 0 \\ x_{k+1} = 2.5x_k - 1.5 & x_k > 0 \end{cases}$$

We apply the graphical method to analyze the properties of the state trajectories, starting from the random initial conditions.



It is possible to verify that, for almost any possible initial conditions, the state trajectories diverge.

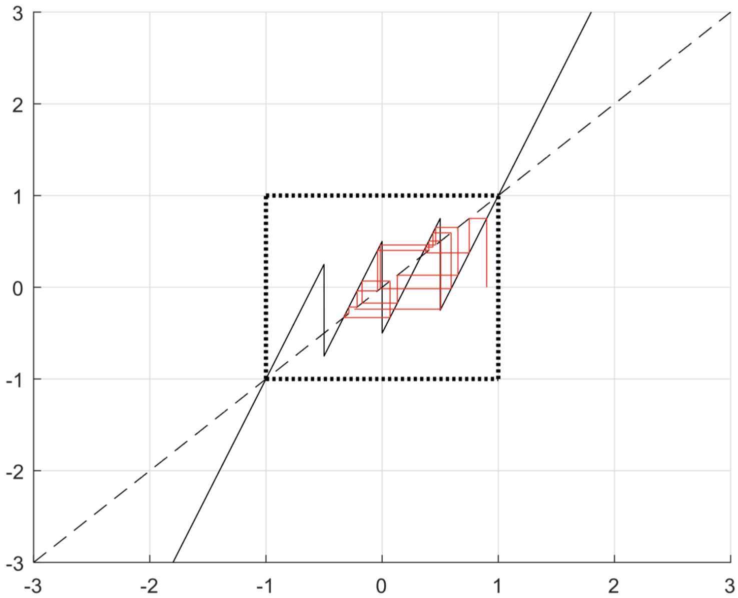
(d) We set $h = \log_e(2.5) = 0.9163$ s, $N_B = 2$. The control law defined in case we operate at the rate limits is

$$u_k = \begin{cases} u_{\max} = 1 & x_k \leq \frac{b}{a}\left(-u_{\max} + \frac{u_{\max}-u_{\min}}{N}\right) = -\frac{1}{2} \\ u_{\max} - \frac{u_{\max}-u_{\min}}{N-1} = \frac{1}{3} & -\frac{1}{2} < x_k \leq \frac{b}{a}\left(-u_{\max} + 2\frac{u_{\max}-u_{\min}}{N}\right) = 0 \\ u_{\max} - 2\frac{u_{\max}-u_{\min}}{N-1} = -\frac{1}{3} & 0 < x_k \leq \frac{b}{a}\left(-u_{\max} + 3\frac{u_{\max}-u_{\min}}{N}\right) = \frac{1}{2} \\ u_{\min} = -1 & x_k > \frac{1}{2} \end{cases}$$

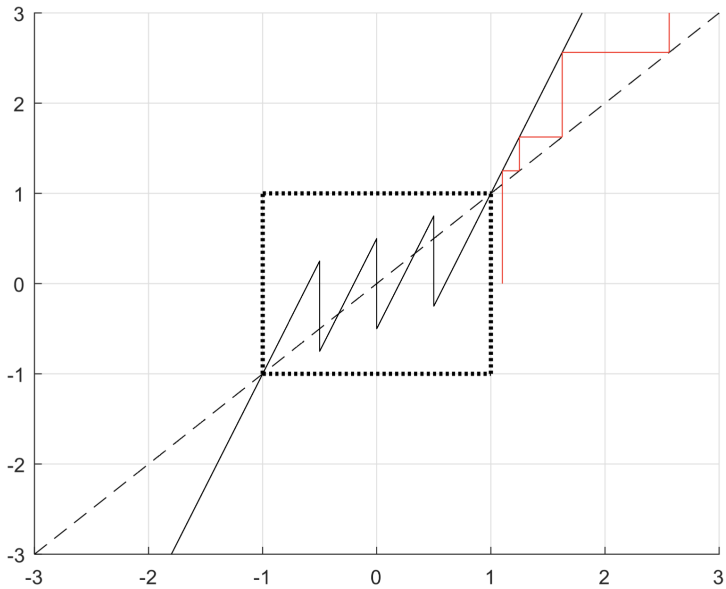
The closed-loop dynamics of the system is

$$\begin{cases} x_{k+1} = 2.5x_k + 1.5 & x_k \leq -\frac{1}{2} \\ x_{k+1} = 2.5x_k + 0.5 & -\frac{1}{2} < x_k \leq 0 \\ x_{k+1} = 2.5x_k - 0.5 & 0 < x_k \leq \frac{1}{2} \\ x_{k+1} = 2.5x_k - 1.5 & x_k > \frac{1}{2} \end{cases}$$

We apply the graphical method to analyze the properties of the state trajectories, starting from random initial conditions.



It is possible to verify that, if $-1 \leq x_0 \leq 1$, the state trajectories remain bounded in the set $[-1, 1]$. On the other hand, if $|x_0| > 1$, the trajectories diverge. This is apparent in the following plot.



2. **Maximal tolerable delay under high network load.** Consider the NCS in Figure 2, where the LTI system is the first-order model

$$\begin{cases} \dot{x} = -x - 5u \\ y = x \end{cases}.$$

Assume that, because of the network load, the nominal sampling time T and network induced delay τ are almost identical, i.e. $T \simeq \tau$.

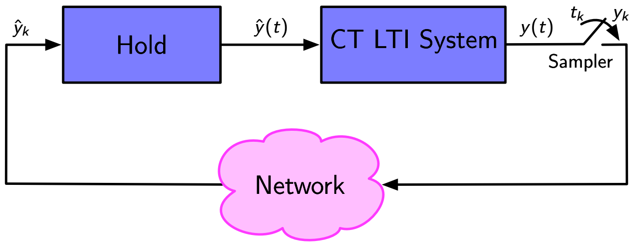


Figure 2: Networked control system

(a) Find by hand all values of T stabilizing the NCS.

Hint: Recall the Jury's criterion: the roots of $\phi(\lambda) = \lambda^2 + \alpha\lambda + \beta$ verify $|\lambda| < 1$ if and only if

$$\begin{array}{l} \beta > -\alpha - 1 \\ \beta > \alpha - 1 \\ \beta < 1 \end{array} .$$

Solution: Given that $A = -1$, $B = -5$, $C = 1$, $T \simeq \tau$ one has $\Gamma(T-\tau) \simeq 0$ and $e^{A(T-\tau)} \simeq 1$. Hence

$$\psi(T,\tau) = \begin{bmatrix} e^{AT} + \Gamma(T-\tau)BC & e^{A(T-\tau)}\Gamma(\tau)B \\ 1 & 0 \end{bmatrix} \simeq \begin{bmatrix} e^{AT} & \Gamma(T)B \\ 1 & 0 \end{bmatrix} .$$

Since $\Gamma(s) = \int_0^s e^{Az}dz = \int_0^s e^{-z}dz = 1 - e^{-s}$, one has

$$\psi(T,\tau) = \begin{bmatrix} e^{-T} & -5(1-e^{-T}) \\ 1 & 0 \end{bmatrix} .$$

By computing the characteristic polynomial of ψ and using Jury's conditions, one has

$$5(1-e^{-T}) > e^{-T} - 1 \tag{1a}$$
$$5(1-e^{-T}) > -e^{-T} - 1 \tag{1b}$$
$$5(1-e^{-T}) < 1 \tag{1c}$$

- From (1c), one obtains $T < |\log(4/5)| = 0.2231$
- Condition (1b) is always verified because $-e^{AT} - 1 < 0$ and $(1-e^{-T}) > 0$ for all $T > 0$
- Condition (1a) is always verified because $-(1-e^{AT}) < 0$ and $(1-e^{-T}) > 0$ for all $T > 0$

Conclusion: $T < |\log(4/5)| = 0.2231$.

(b) The plot in Figure 3, seen in the lectures, represents the pairs (T,τ) verifying $T > \tau$ and yielding asymptotic stability (green points) or not (red points) of the NCS. Identify the pair $(\bar{T},\tau)$ corresponding to maximum allowable sampling period $\bar{T}$ found in the previous point.

Solution: It is the point on the boundary $T = \tau$ with $T = \bar{T} = 0.2231$.

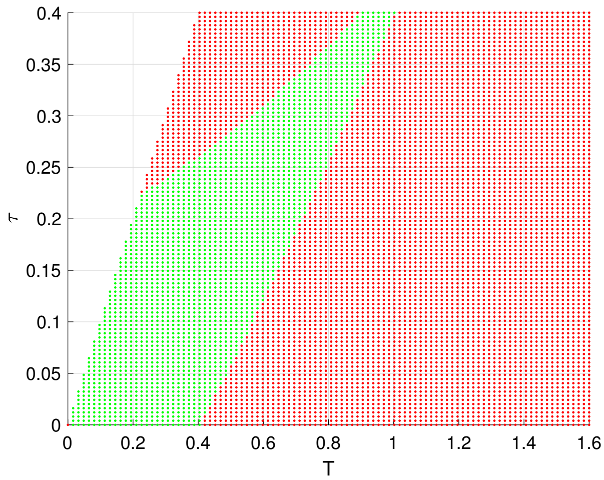


Figure 3: Region of stability for the system in Exercise 2