Networked Control Systems (ME-427)- Exercise session 9

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1. **Bounded evolution for averaging systems** Consider the fictitious train map given in Figure 1.

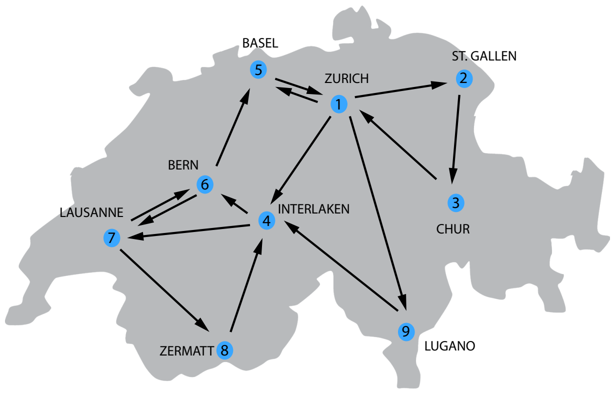


Figure 1: Fictitious map of trains in Switzerland.

(a) Write the unweighted adjacency matrix of this graph. By using the theory developed in class and Matlab answer the following questions.

i. Is the graph strongly connected? What does this mean for a passenger?

ii. Is the graph acyclic?

iii. What is the number of links of the shortest path connecting St. Gallen to Zermatt?

iv. Is it possible to go from Bern to Chur using 4 links? And 5?

v. How many different routes, with strictly less then 9 links, start from Zurich and end in Lausanne? (You may even pass by Zurich and Lausanne more then once during your trip)

Solution:

(a) The adjacency matrix of the graph is given as

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

i. Compute $\bar{A} = \sum_{i=1}^{8} A^i$. If the sum $\bar{A} + I$ results in a positive matrix, then the graph is strongly connected. For the passenger, this implies that any town can be reached from any other.

ii. The graph is not acyclic because $\bar{A}$ has a positive element on the diagonal (in fact, they are all positive).

iii. St. Gallen and Zermatt correspond to nodes 2 and 8 respectively. We look at the element $(2,8)$ in matrices A^k, $k = 1, 2, \ldots$. The desired $\bar{k}$ is the first value of k for which the element $(2,8)$ is positive.

iv. As seen in the lecture, the (i,j) entry of A^k equals the number of directed paths of length k (including paths with self-loops) from node i to node j. Check the (6,3) element of A^4 and A^5. It turns out that it is possible to reach Chur from Bern using 4 links but not 5.

v. Set $\bar{A} = \sum_{i=1}^{8} A^i$. The value of $\bar{A}_{17}$ gives the desired number. Between Zurich and Lausanne, the number of different routes possible with less than 9 links is 79.

2. **Simple properties of stochastic matrices** [Textbook E2.1] Let $A_1, A_2, \ldots, A_k$ be $n \times n$ matrices and $A_1 A_2 \ldots A_k$ be their product. Show that

   (a) if $A_1, A_2, \ldots, A_k$ are non-negative, then their product is non-negative,

   (b) if $A_1, A_2, \ldots, A_k$ are row-stochastic, then their product is row-stochastic, and

   (c) if $A_1, A_2, \ldots, A_k$ are doubly-stochastic, then their product is doubly stochastic.

   Solution:

   (a) From the matrix product formula: $(A_1 A_2)_{ij} = \sum_k (A_1)_{ik} (A_2)_{ik}$, we know that the product of two non-negative matrices is non-negative. Recursively we know that the product of multiple non-negative matrices is non-negative.

   (b) Note that $A_1 A_2 \ldots A_k \mathbb{1}_n = A_1 A_2 \ldots A_{k-1} \mathbb{1}_n = \cdots = A_1 \mathbb{1}_n$, because every matrix A_i is row-stochastic. It follows that the product $A_1 A_2 \ldots A_k$ is also row-stochastic.

   (c) Assume $A_1, A_2, \ldots, A_k$ are doubly-stochastic. If one left multiplies $A_1 A_2 \ldots A_k$ by $\mathbb{1}_n^T$, then, following the same proof as from fact (b), one can show that this product is column-stochastic.

3. **Powers of primitive matrices** [Textbook E2.5] Let $A \in \mathbb{R}^{n \times n}$ be non-negative. Show that $A^k \succ 0$, for some $k \in \mathbb{N}$, implies $A^m \succ 0$ for all $m \geq k$.

   Hint: If column j of A is zero, then column j of $A^k, k > 0$ is zero. This fact has a simple proof and can be useful for solving the exercise.

   Solution: Note that $(A)_{ij}^{k+1} = (A^k A)_{ij} = \underbrace{\begin{bmatrix} \tilde{a}_{i1} \cdots \tilde{a}_{in} \end{bmatrix}}_{\text{Row } i \text{ of } A^k} \underbrace{\begin{bmatrix} a_{1j} \\ \vdots \\ a_{nj} \end{bmatrix}}_{\text{Column } j \text{ of } A}$. If at least one $a_{lj} > 0, l = 1, \ldots, n$, then $(A)_{ij}^{k+1} > 0$. $(A)_{ij}^{k+1}$ is only zero if $\begin{bmatrix} a_{1j} \\ \vdots \\ a_{nj} \end{bmatrix} = 0$. But if this happens, one cannot have $A^k \succ 0$. Indeed, column j of A^2 will be zero as well. The same holds for A^k, for all $k > 0$.

4. **On some non-negative matrices** [Textbook E2.9] How many 2×2 matrices exist that are simultaneously doubly stochastic, irreducible and not primitive? Justify your claim.

   Hint: Parametrize the entry in position $(1,1)$ as $x \in [0,1]$.

   Solution: Any 2×2, doubly-stochastic matrix A_x must satisfy

$$A_x = \begin{bmatrix} x & 1-x \\ 1-x & x \end{bmatrix} \tag{1}$$

   Let us analyze three different cases:

   - $x = 1 \Longrightarrow A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, which is not irreducible, because it is an upper triangular matrix,
   - $0 < x < 1 \Longrightarrow [A_x]_{ij} > 0$ for every $i, j \in \{1, 2\}$, hence A_x is primitive, and

- $x = 0 \Longrightarrow A_0 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, which is irreducible, but not primitive, because

$$[A_0]^{2k} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } [A_0]^{2k+1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

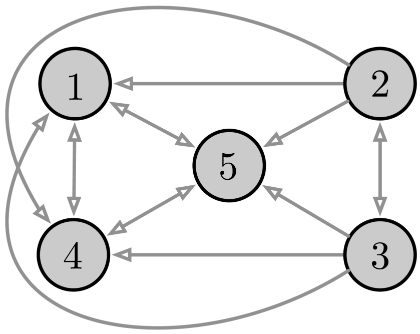
As a conclusion, the only 2×2, doubly stochastic, irreducible and not primitive matrix is $A_0 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

5. **An example reducible or irreducible matrix.** [Textbook E4.7] Consider the binary matrix:

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Prove that A is irreducible or prove that A is reducible by providing a permutation matrix P that transforms A into an upper block-triangular matrix, i.e., $P^T AP = \begin{bmatrix} \star & \star \\ \mathbf{0}_{(n-r)\times r} & \star \end{bmatrix}$ for some r such that $0 < r < n = 5$.

Solution: It can be seen that A is reducible (i.e., not irreducible), since the induced graph features two strongly connected components consisting of the node set $V_1 = \{2, 3\}$ and $V_2 = \{1, 4, 5\}$, and the set V_1 is not reachable from V_2.



Accordingly, we can find a permutation matrix P whose first two columns correspond to V_1 and the rest belongs to V_2:

$$P^\top = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{Check: } P^\top \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \\ 4 \\ 5 \end{bmatrix}$$

By means of this permutation matrix, the A matrix can be put into upper block-triangular form:

$$\underbrace{\begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}}_{P^\top} \underbrace{\begin{bmatrix} 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}}_{P} = \left[\begin{array}{cc|ccc} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ \hline 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right].$$