NAME: ______________________ **SCIPER:** ____________

This exam is open book and open notes, but no computers are allowed.

Please answer all questions. Values of each question are given below.

Problem:	1	2	3	4	5	6	Total
Value:	25	15	15	15	15	15	100
Grade:							

Problem 1.

When there are multiple choices in the following, select all statements that are true.

a) Consider the linear system $x^+ = \begin{bmatrix} 0.9 & 0 \\ 0 & 1.7 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$ with the output $y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$. Which of the following statements are true?

- ◯ The system is controllable
- ◯ The system is observable
- ◯ With the control law $u = \begin{bmatrix} 1.8 & -5.6 \end{bmatrix} x$, the closed-loop system is stable
- ◯ With the control law $u = \begin{bmatrix} 1.8 & -5.6 \end{bmatrix} x$, $V(x) = x^T \begin{bmatrix} 0.9 & 0 \\ 1 & -1.6 \end{bmatrix} x$ is a Lyapunov function for the closed-loop system
- ◯ None of the above

b) Consider the set $P = \cap_{i=1\ldots N} H_i$, where each H_i is a halfspace and N is finite. Which of the following statements are true?

- ◯ $\max\ a^T x$ s.t. $x \in P$ is finite for all a
- ◯ P is a polyhedron
- ◯ P is a polytope
- ◯ P is a convex set
- ◯ P may be non-convex
- ◯ $\max\ x^T x$ s.t. $x \in P$ is a convex problem

c) Which of the following sets are convex?

- ◯ A slab: $\{x \in \mathbb{R}^n \mid \alpha \leq a^T x \leq \beta\}$
- ◯ A hypercube: $\{x \in \mathbb{R}^n \mid \alpha_i \leq x_i \leq \beta_i,\ i = 1, \ldots, n\}$
- ◯ The intersection $S = S_1 \cap S_2$ of two convex sets S_1 and S_2

d) Which of the following functions are convex?

- ◯ The quadratic function: $f(x) = x^T \begin{bmatrix} 1.5 & 0 \\ 0 & -0.2 \end{bmatrix} x$
- ◯ The function: $f(x) = x^4$
- ◯ The l_p norm: $f(x) = \|x\|_p$, with $p \geq 1$
- ◯ The indicator function on a convex set $\mathbb{C}$: $f(x) = \begin{cases} 0 & x \in \mathbb{C} \\ \infty & \text{otherwise} \end{cases}$

e) Consider the problem $d(\lambda) = \min_x \ a \cdot x^2 + (x-1)\lambda$. Mark all correct statements

- ◯ d is convex
- ◯ d is concave
- ◯ The convexity of d depends on a
- ◯ d is neither convex nor concave

f) Given the function $f(x) = \begin{cases} \infty & |x| > 1 \\ 0 & |x| \leq 1 \end{cases}$, what is $\mathrm{prox}_{f,\rho}(v)$

- ◯ $\mathrm{prox}_{f,\rho}(v) = 0$
- ◯ $\mathrm{prox}_{f,\rho}(v) = \infty$
- ◯ $\mathrm{prox}_{f,\rho}(v) = v$
- ◯ $\mathrm{prox}_{f,\rho}(v) = \begin{cases} 1 & v \geq 1 \\ -1 & v \leq -1 \\ v & \text{otherwise} \end{cases}$
- ◯ $\mathrm{prox}_{f,\rho}(v) = \begin{cases} 1 & 1 \leq v \leq 1 + \sqrt{2/\rho} \\ -1 & -1 - \sqrt{2/\rho} \leq v \leq -1 \\ v & \text{otherwise} \end{cases}$

g) The maximum control invariant set of a linear system subject to polyhedral constraints is

- ◯ Convex
- ◯ Polytopic
- ◯ None of the above

h) Let S be an invariant set for the linear system $x^+ = Ax$. For which values of α is αS also an invariant set for this system?

- ◯ $\alpha = 0$
- ◯ $0 < \alpha < 1$
- ◯ $\alpha > 1$
- ◯ $\alpha = 1$

i) Let $x^+ = f(x, u)$ be a system with constraints $(x, u) \in X \times U$, and let C be the maximal control invariant set of the system. Mark the true statements.

- ◯ There exists an $x \in X$ and a $u \in U$ such that $f(x, u) \in C$
- ◯ For all $x \in X$, there exists a $u \in U$ such that $f(x, u) \in C$
- ◯ There exists an $x \in X \backslash C$ and a $u \in U$ such that $f(x, u) \in C$
- ◯ There does not exist an $x \in X \backslash C$, and a $u \in U$ such that $f(x, u) \in C$

j) Let $x^+ = f(x, u)$ be a system with constraints $(x, u) \in X \times U$, and let X_∞ be the maximal invariant set of the system $x^+ = f(x, \kappa(x))$ for some controller $\kappa(x)$. Mark the true statements.

- ◯ For all $x \in X$, there exists a $u \in U$ such that $f(x, u) \in X_\infty$
- ◯ For all $x \in X_\infty$, there exists a $u \in U$ such that $f(x, u) \in X_\infty$
- ◯ It is possible that there exists an $x \in X \backslash X_\infty$ and $u \in U$ such that $f(x, u) \in X_\infty$
- ◯ For all $x \in X \backslash X_\infty$, there doesn't exist any $u \in U$ such that $f(x, u) \in X_\infty$

k) If C is a control invariant set for the system $x^+ = f(x, u)$, then C is a positively invariant set for the autonomous system $x^+ = f(x, \kappa(x))$, under which of the following control laws

- ◯ $\kappa(x) = 0$
- ◯ $\kappa(x) = Kx$, where K is a stabilizing control gain
- ◯ $\kappa(x) = \text{argmin}\{\|u\|_2\}$
- ◯ $\kappa(x) = \text{argmin}\{\|u\|_2 \mid f(x, u) \in C\}$
- ◯ all of the above
- ◯ none of the above

l) A linear autonomous system with convex constraints has two invariant sets S_1 and S_2. Indicate which of the following statements hold

- ◯ $S_1 \cap S_2$ is an invariant set of the system
- ◯ $S_1 \cup S_2$ is an invariant set of the system
- ◯ The convex hull of $S_1 \cup S_2$ is an invariant set of the system

m) Given a linear system $x^+ = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$ and a state-feedback control law $u = \begin{bmatrix} -3 & 1 \end{bmatrix} x$, which of the following is a Lyapunov function for the closed-loop system $x^+ = (A + BK)x$?

- ◯ $x^T x$
- ◯ $\frac{\|x+3\|_\infty}{x^T x}$
- ◯ $x^T \begin{bmatrix} 1 & 0.7 \\ -0.3 & 0.9 \end{bmatrix} x$
- ◯ None of the above

n) Consider the following four MPC problems.

$$
\begin{aligned}
J_1(x) = \min \; & \sum_{i=0}^{\infty} x_i^T Q x_i + u_i^T R u_i \\
\text{s.t.} \quad & x_{i+1} = A x_i + B u_i \\
& (x_i, u_i) \in X \times U \\
& x_0 = x
\end{aligned}
$$

$$
\begin{aligned}
J_2(x) = \min \; & \sum_{i=0}^{N-1} x_i^T Q x_i + u_i^T R u_i + J_1(x_N) \\
\text{s.t.} \quad & x_{i+1} = A x_i + B u_i \\
& (x_i, u_i) \in X \times U \\
& x_0 = x
\end{aligned}
$$

$$
\begin{aligned}
J_3(x) = \min \; & \sum_{i=0}^{\infty} x_i^T Q x_i + u_i^T R u_i \\
\text{s.t.} \quad & x_{i+1} = A x_i + B u_i \\
& x_0 = x
\end{aligned}
$$

$$
\begin{aligned}
J_4(x) = \min \; & \sum_{i=0}^{N-1} x_i^T Q x_i + u_i^T R u_i + J_3(x_N) \\
\text{s.t.} \quad & x_{i+1} = A x_i + B u_i \\
& (x_i, u_i) \in X \times U \\
& x_N \in X_f \\
& x_0 = x
\end{aligned}
$$

with $Q \succ 0$, $R \succ 0$ and $X_f \subseteq X$ an invariant set for the system $x^+ = Ax + B\kappa(x)$, where $\kappa(x)$ is the control law defined by the MPC problem on the bottom left. Mark all correct statements.

- ◯ $J_1(x) \leq J_2(x)$
- ◯ $J_1(x) \geq J_2(x)$
- ◯ $J_1(x) = J_2(x)$
- ◯ $J_3(x) \leq J_1(x)$
- ◯ $J_3(x) \geq J_1(x)$
- ◯ $J_3(x) = J_1(x)$
- ◯ $J_3(x) \leq J_4(x)$
- ◯ $J_3(x) \geq J_4(x)$
- ◯ $J_3(x) = J_4(x)$

o) Consider the uncertain linear system $x^+ = Ax + Bu + w$ and two disturbance sets W_1 and W_2 such that $W_1 \subset W_2$. Which of the following statements is true for a polytopic set Ω

- ◯ $pre^{W_1}(\Omega) \supseteq pre^{W_2}(\Omega)$
- ◯ $pre^{W_1}(\Omega) \subseteq pre^{W_2}(\Omega)$
- ◯ Nothing can be said without knowledge of the matrices A and B

p) Consider a linear system $x^+ = Ax + Bu$ and the MPC controller

$$\begin{aligned} J^\star(x) = \min \ & \sum_{i=0}^{N-1} l(x_i, u_i) \\ \text{s.t.} \quad & x_{i+1} = Ax_i + Bu_i \\ & (x_i, u_i) \in (X \times U) \\ & x_0 = x \end{aligned}$$

When running the controller, we observe that the system is subject to a lot of noise, and that the optimization problem is sometimes infeasible. Which of the following methods will ensure recursive feasibility?

- ◯ Use a longer horizon
- ◯ Add a terminal constraint $x_N \in X_f$
- ◯ Add a terminal cost $V_f(x_N)$
- ◯ Use a tracking MPC formulation
- ◯ Use a soft-constrained MPC formulation

q) Consider the following MPC controller

$$\begin{aligned} J^\star(x) = \min \ & \sum_{i=0}^{N-1} q^T x_i + r^T u_i + p^T x_N \\ \text{s.t.} \quad & x_{i+1} = Ax_i + Bu_i \\ & (x_i, u_i) \in (X \times U) \\ & x_N = 0 \\ & x_0 = x \end{aligned}$$

where $0 \in U$.

- ◯ The MPC controller is recursively feasible
- ◯ The MPC controller will asymptotically stabilize the system $x^+ = Ax + Bu$
- ◯ Neither of the above

r) Consider the standard (top) and soft-constrained (bottom) MPC problem formulations

Problem 1

$$
\begin{aligned}
J^\star(x) \quad = \min_{u,x} \quad & \sum_{i=0}^{N-1} x_i^T Q x_i + u_i^T R u_i + x_N^T P x_N \\
\text{s.t.} \quad & x_{i+1} = Ax_i + Bu_i \\
& Gx_i \leq g \\
& G_N x_N \leq g_N \\
& Hu_i \leq h \\
& x_0 = x
\end{aligned}
$$

Problem 2

$$
\begin{aligned}
J^\star_{soft}(x) \quad = \min_{u,x,\epsilon} \quad & \sum_{i=0}^{N-1} x_i^T Q x_i + u_i^T R u_i + x_N^T P x_N + \rho \sum_{i=0}^{N} \epsilon_i^T \epsilon_i \\
\text{s.t.} \quad & x_{i+1} = Ax_i + Bu_i \\
& Gx_i \leq g + \epsilon_i \\
& G_N x_N \leq g_N + \epsilon_N \\
& Hu_i \leq h \\
& x_0 = x \\
& \epsilon_i \geq 0
\end{aligned}
$$

Mark all the correct statements

- ◯ $J^\star_{soft}(x) \leq J^\star(x)$ for all x feasible for both problems
- ◯ $J^\star_{soft}(x) \geq J^\star(x)$ for all x feasible for both problems
- ◯ $J^\star_{soft}(x) \geq J^\star(x)$ if ρ is sufficiently large
- ◯ For a given x, Problem 1 is feasible if Problem 2 is feasible
- ◯ For a given x, Problem 2 is feasible if Problem 1 is feasible

Problem 2.

/15

Consider the linear system $x^+ = Ax$ subject to the constraints $x \in X$. Let $Y \subseteq X$ and $Z \subseteq X$ be invariant sets for this system and $x_s \in \operatorname{int} X$[1] a steady-state solution, $x_s = Ax_s$.

1. Prove that the set $\bar{Y} = Y \oplus \{x_s\} = \{x + x_s \mid x \in Y\}$ is also an invariant set for this system if $\bar{Y} \subseteq X$

2. Prove that there exists a scaling factor $\lambda > 0$ such that $\{x_s\} \oplus \lambda Z := \{\lambda x + x_s \mid x \in Z\}$ is an invariant set for the system
(Note that $\{x_s\} \oplus Z$ may, or may not be a subset of X)

[1]Note : $\operatorname{int} X := \{x \mid \exists \epsilon > 0\,, x + y \in X,\ \forall y \in \mathbb{B}_\epsilon\}$ refers to the interior of X, where $\mathbb{B}_\epsilon := \{y \mid \|y\| \leq \epsilon\}$

3. Let $A = 1$, $X = [-2, 2]$ and $Z = [-1, 1]$. Compute the largest λ as a function of the steady-state x_s

Problem 3. /15

Consider the linear discrete time system

$$x^+ = 1.1x + u + w$$

where the state x, the input u and the disturbance w, are all of dimension one and are constrained as follows

$$X = [-2, 2] \qquad U = [-0.8, 0.8] \qquad W = [-0.2, 0.1]$$

Your goal is to design a tube-based MPC controller for this system.

1. Compute the minimal robust invariant set $\mathcal{E}$ for this system with the control law $u = Kx$, for $K = -0.3$

   Hint : $[a, b] \oplus [c, d] = [a + c, b + d]$

2. Compute the tightened constraints $\tilde{X} = X \ominus \mathcal{E}$ and $\tilde{U} = U \ominus K\mathcal{E}$

   Hint : $[a, b] \ominus [c, d] = [a - c, b - d]$

3. Verify that the set $\mathcal{X}_f = [-1, 1.5]$ if used as a terminal constraint for the tube-based MPC scheme will result in a recursively feasible controller for the terminal control law $u = -0.3x$

Problem 4.

/15

Consider the following parametric QP problem with the parameter x.

$$\begin{aligned} f^\star(x) = \min_z \; & z^2 + (x+1)z + x \\ \text{s.t.} \; & z \geq 2x \\ & z \geq 0 \end{aligned}$$

1. Give matrices M, Q and vector q such that the optimal solution of the problem above is a linear transformation of the solution $y(x)$ to the following parametric LCP

$$w - My = Qx + q \qquad w, y \geq 0 \qquad w^T y = 0 \tag{1}$$

2. Draw the complementarity cones of the pLCP (1)

3. Compute the optimal value function $f^\star(x)$

   Write the answer here:

$$f^\star(x) = \begin{cases} \rule{4cm}{0.4pt} & x \in [\_\_\_\_, \_\_\_\_] \\ \rule{4cm}{0.4pt} & x \in [\_\_\_\_, \_\_\_\_] \\ \rule{4cm}{0.4pt} & x \in [\_\_\_\_, \_\_\_\_] \end{cases}$$

Problem 5.

/15

Consider the linear system $x^+ = Ax + Bu$ and the following unconstrained MPC problem

$$\begin{aligned} \min \ & \sum_{i=0}^{N-1} x_i^T Q x_i + u_i^T R u_i + x_N^T S x_N \\ \text{s.t.} \ & x_{i+1} = Ax_i + Bu_i \\ & x_0 = x \end{aligned}$$

1. Assume that $\rho(A) < 1$, $Q \succ 0$ and $R \succ 0$.
   Give a sufficient condition on the matrix S that ensures stability of the closed-loop system with the terminal control law $u = 0$.

2. Assume that $\rho(A) > 1$, $Q \succ 0$ and $R \succ 0$.
   Give a condition on the matrix S such that the resulting MPC control law is equal to the infinite-horizon LQR solution. (You can use any terminal control law you like)

3. Assume that $Q = 0$, $R = 1$ and $B = 1$.

i) Use dynamic programming to compute the MPC control law for $A = -0.5$, $S = -0.5$ and $N = 2$.

ii) Is the resulting closed-loop system stable, or unstable?

Problem 6.

/15

Consider the convex optimization problem

$$\begin{aligned} &\min\ x^T Q x + q^T x + r \\ &\text{s.t. } \|Px - p\|_2^2 \le 1 \end{aligned} \tag{2}$$

1. Derive a closed-form expression for the proximal operator of the function

$$f(y) = \begin{cases} 0 & \|y\|_2^2 \le 1 \\ \infty & \text{otherwise} \end{cases}$$

   Hint : Recall the definition of the proximal operator

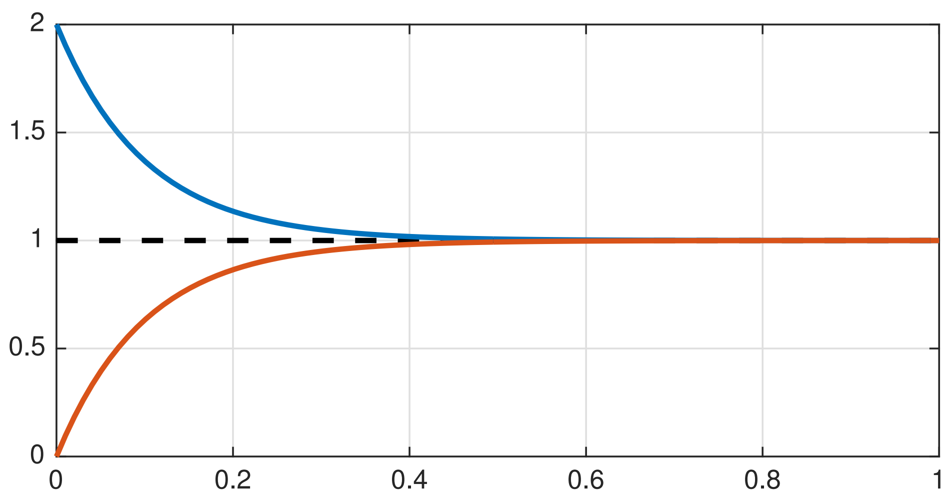
$$\text{prox}_{f,\rho}(v) := \text{argmin}_y\ \ f(y) + \frac{\rho}{2}\|y - v\|_2^2$$

2. Give functions f, g and matrices A, B and b so that problem (3) is equivalent to (2).

$$\begin{aligned} &\min\ f(x) + g(y) \\ &\text{s.t. } Ax + By = b \end{aligned} \tag{3}$$

3. Give the three steps of the ADMM algorithm for this problem, using the prox operator that you derived in Part 1.

4. Suppose that we solve problem (2) using (a) A logarithmic barrier method and (b) ADMM. The figure below shows the value of the objective function during the optimization for each case, with the optimal value marked by a dashed line.



Which line corresponds to ADMM? Explain your answer.