**NAME:** ______________________ **SCIPER:** ____________

This exam is open book and open notes, but no computers are allowed.

Please answer all questions. Values of each question are given below.

| Problem: | 1 | 2 | 3 | 4 | 5 | 6 | Total |
|---|---|---|---|---|---|---|---|
| Value: | 20 | 20 | 15 | 15 | 15 | 15 | 100 |
| Grade: | | | | | | | |

## Problem 1.

When there are multiple choices in the following, select all statements that are true.

a) Is the union of a finite set of ellipses convex?

- ○ Yes
- ○ No
- ⊗ Not enough information

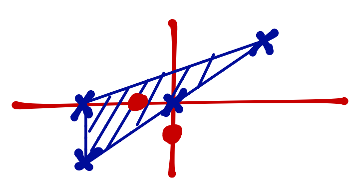
b) Is the intersection of an ellipse and a polytope convex?

- ⊗ Yes
- ○ No
- ○ Not enough information

c) If $f$ is a convex function and $x$ is a point such that $\nabla f(x) = 0$, then necessarily $x$ is a

- ⊗ local minimum of $f$
- ⊗ global minimum of $f$
- ⊗ global minimum of $f$ if $\nabla^2 f(x) \succ 0$
- ○ global minimum of $f$ if $\nabla^2 f(x) \prec 0$

d) Let the set $S$ be $\{(1,1), (-1,0), (0,0), (-1,-1)\}$.



- ⊗ The point $(-1/3, 0)$ is contained in the convex hull of $S$
- ○ The point $(0, -1/3)$ is contained in the convex hull of $S$

e) Consider the nominal system $x^+ = f(x)$ and let $S$ be a non-empty set such that $S \subseteq \text{pre}(S)$. Under which of the following conditions is $S$ an invariant set of the uncertain system $y^+ = f(y) + w$ where $w$ is restricted to lie in $W$? (Note that the pre-set operator below refers to the nominal system)

- ☒ $S \subseteq \text{pre}(S \ominus W)$ $= \text{pre}^W(S)$
- ◯ $S \oplus W \subseteq \text{pre}(S)$
- ◯ $S \subseteq \text{pre}(S)$
- ◯ $S \subseteq \text{pre}(S) \oplus W$

(Recall that $\ominus$ is the Pontryagin difference, and $\oplus$ is the Minkowski sum.)

f) The function $\cosh x = (e^x + e^{-x})/2$ is

$f(x) = \frac{e^x + e^{-x}}{2}$ $\quad \frac{df}{dx} = \frac{e^x - e^{-x}}{2}$

$\frac{d^2 f}{dx^2} = \frac{e^x + e^{-x}}{2} > 0 \ \forall x$

- ☒ Convex
- ◯ Concave
- ◯ Neither convex nor concave
- ◯ Affine

g) On the domain $y > 0$, the function $-x^2/y$ is

$\nabla = \begin{bmatrix} -2x/y \\ x^2/y^2 \end{bmatrix}$ $\quad \nabla^2 = \begin{bmatrix} -2/y & 2x/y^2 \\ 2x/y^2 & -2x^2/y^3 \end{bmatrix}$

$= \frac{-2}{y^3}\begin{bmatrix} y^2 & xy \\ xy & x^2 \end{bmatrix} = \underbrace{\frac{-2}{y^3}}_{<0} \underbrace{\begin{pmatrix} y \\ x \end{pmatrix}'\begin{pmatrix} y \\ x \end{pmatrix}}_{>0}$

- ◯ Convex
- ☒ Concave
- ◯ Neither convex nor concave
- ◯ Affine

h) The function $\max\{x + 4,\ 2x\}$ is

- ☒ Convex
- ◯ Concave
- ◯ Neither convex nor concave
- ◯ Affine

i) Consider the system $x^+ = -0.5x$. Which of the following sets are invariant?

- ☒ $\{x \mid x^4 \leq 5\}$
- ◯ $\{x \mid x^3 \leq 5\}$
- ☒ $\{x \mid -1 \leq x \leq 2\}$
- ◯ $\{x \mid -1/2 \leq x \leq 2\}$

j) Consider the system

$$\begin{pmatrix} x_1^+ \\ x_2^+ \end{pmatrix} = \begin{pmatrix} 0.5x_1 \\ 1.5x_2 \end{pmatrix}$$

Which of the following sets is invariant?

- ☒ $\{x \mid x_2 = 0,\ x_1 \le 10\}$
- ◯ $\{x \mid x_1 = 0,\ x_2 \le 10\}$
- ◯ $\{x \mid x_2 = x_1\}$

k) Consider the following predictive control problem, which defines the receding horizon control law $\pi(x)$

$$\begin{aligned} \min\ & \sum_{i=0}^{N} x_i^T Q x_i + u_i^T R u_i \\ \text{s.t. } & x_{i+1} = Ax_i + Bu_i \\ & x_0 = x \end{aligned}$$

Which of the following statements is true:

- ◯ The control law $\pi(x)$ is quadratic
- ☒ The control law $\pi(x)$ is linear
- ◯ If the closed-loop system is stable for $N = 5$, then it will be stable for $N = 6$
- ◯ The closed-loop system will be stable if $x^+ = Ax$ is unstable and $N \ge \text{rank}(A)$

l) Consider the following predictive control problem,

$$\begin{aligned} \min\ & \sum_{i=0}^{N} x_i^T Q x_i + u_i^T R u_i \\ \text{s.t. } & x_{i+1} = Ax_i + Bu_i \\ & x_i \in X,\ \ u_i \in U \\ & x_0 = x \end{aligned}$$

Which of the following statements is true:

- ◯ The feasible set of the above optimization problem is larger for $N = 5$, than for $N = 4$
- ◯ If the closed-loop system is stable for $N = 5$, then it will be stable for $N = 6$
- ◯ If the MPC problem is recursively feasible for $N = 5$, then is will be for $N = 6$
- ☒ None of the above

m) What is the proximal operator $\text{prox}_{f,\rho}(v)$ of the function $f(x) = \|Ax - b\|_2^2$?

- ○ $v$
- ○ $Av - b$
- ○ $(A^T A + \rho I)^{-1}(A^T b + v)$
- ☒ $(2A^T A + \rho I)^{-1}(2A^T b + \rho v)$

n) What is the proximal operator $\text{prox}_{f,\rho}(v)$ of the function

$$f(x) = \begin{cases} 0 & x = 0 \\ 0 & x = 1 \\ \infty & \text{otherwise} \end{cases}$$

- ○ $\rho v$
- ○ $v$
- ☒ $\begin{cases} 0 & \text{if } v \leq 0.5 \\ 1 & \text{if } v > 0.5 \end{cases}$
- ○ $\infty$

o) Let $f(x)$ be the indicator function for the convex set $C$. What is $\text{prox}_{f,\rho}(v)$ for a point $v \in C$?

- ○ $\rho v$
- ☒ $v$
- ○ $\rho C$
- ○ $\infty$

p) Consider the standard (right) and soft-constrained (left) MPC problem formulations:

$$
\begin{aligned}
J^\star_{soft}(x) = \min_u \ & \sum_{i=0}^{N-1} l(x_i, u_i) + V_N(x_N) + \rho \sum_{i=0}^{N-1} \epsilon_i^T \epsilon_i \\
\text{s.t.} \ & x_{i+1} = Ax_i + Bu_i \\
& Gx_i \le g + \epsilon_i \\
& Hu_i \le h \\
& \epsilon_i \ge 0
\end{aligned}
\qquad\Bigg|\qquad
\begin{aligned}
J^\star(x) = \min_u \ & \sum_{i=0}^{N-1} l(x_i, u_i) + V_N(x_N) \\
\text{s.t.} \ & x_{i+1} = Ax_i + Bu_i \\
& Gx_i \le g \\
& Hu_i \le h
\end{aligned}
$$

where the standard problem has been designed with appropriate terminal weights and constraints so that the resulting problem is recursively feasible and $J^\star(x)$ is a Lyapunov function. Let $Z$ be the set of states for which the standard problem is feasible, and $\pi_{soft}(x)$ the control law resulting from solving the soft-constrained problem. Which of the following conditions will be satisfied:

- [x] $J^\star_{soft}(x) \le J^\star(x)$ for all $x \in Z$
- [ ] $J^\star_{soft}(Ax + B\pi_{soft}(x)) \le J^\star_{soft}(x)$ for all $x \in Z$
- [ ] $J^\star_{soft}(Ax + B\pi_{soft}(x)) \le J^\star_{soft}(x)$ for all $x \notin Z$

q) Consider the MPC control law for the linear system $x^+ = Ax + Bu$

$$
\begin{aligned}
\min \ & \sum_{i=0}^{N} l(x_i, u_i) + x_N^T P x_N \\
\text{s.t.} \ & x_{i+1} = Ax_i + Bu_i \\
& x_i \in X, \ \ u_i \in U \\
& x_N \in X_f \\
& x_0 = x
\end{aligned}
$$

Where $K$ is a matrix such that the set $X_f \subset X$ is invariant for $x^+ = (A + BK)x$, $KX_f \subset U$, and the function $l(x, u)$ is positive definite. Which of the following additional conditions will ensure asymptotic stability of the closed-loop system?

- [ ] $(A + BK)^T P (A + BK) - P \preceq 0$
- [x] $x^T[(A + BK)^T P (A + BK) - P]x \le -l(x, Kx)$ for all $x \in X_f$
- [ ] $X_f$ is a control invariant set

r) Which of the following statements implies that $S = \left\{x \,\middle|\, x^T P x \le 1\right\}$, $P \succeq 0$ is an invariant set for the system $x^+ = Ax$?

- [ ] $A^T P A \succeq P$
- [x] $A^T P A \preceq P$
- [ ] $A^T P A \succeq 0$
- [ ] $A^T P A \preceq 0$

**Problem 2.**

/20

Consider the following quadratically constrained quadratic program:

$$\begin{aligned} &\min \frac{1}{2}x^T Q x + c^T x \\ &\text{s.t. } x^T x \le \alpha \end{aligned} \tag{1}$$

where $Q \succ 0$ is a positive definite matrix.

a) Barrier method

i) Consider the barrier function $\phi(x) = -\sum_{i=1}^{m} \log(-g_i(x))$, where $g_i(x) \le 0$ are the constraints of the problem. Compute the function $\phi(x)$, its gradient and its hessian for the optimization problem given above.

ii) Compute the Newton direction for solving the centering step of the barrier interior-point method for the above problem.

iii) Let $Q = I$, $c = \begin{pmatrix} 1 & 2 \end{pmatrix}^T$ and $\alpha = 3$. (1) Compute the Newton direction at the point $x = \begin{pmatrix} 1 & 1 \end{pmatrix}^T$ for a value of the barrier parameter $\kappa = 1$ and (2) demonstrate that the result is a decent direction.

i/ $g(x) = x'x - \alpha \qquad \nabla g(x) = 2x \qquad \nabla^2 g(x) = 2I$

$\phi(x) = -\log(-x'x + \alpha) \qquad \nabla\phi(x) = \frac{2x}{\alpha - x'x} \qquad \nabla^2\phi(x) = \frac{4xx'}{(\alpha - x'x)^2} + \frac{2I}{\alpha - x'x}$

ii/ $\Delta z_{nt} = (\nabla^2 f(x) + \kappa\nabla^2\phi(x))^{-1} \cdot (-\nabla f(x) - \kappa\nabla\phi(x))$

$= \left(Q + \frac{\kappa 4}{(\alpha - x'x)^2} xx' + \frac{2\kappa}{\alpha - x'x} I\right)^{-1} \left(-Qx - c - \frac{\kappa 2}{\alpha - x'x} x\right)$

iii/ (1) $\Delta z_{nt} = \left[I + 4\binom{1}{1}\binom{1}{1}' + 2I\right]^{-1}\left(-\binom{1}{1} - \binom{1}{2} - 2\binom{1}{1}\right)$

$= \begin{bmatrix} 7 & 4 \\ 4 & 7 \end{bmatrix}^{-1}\left(-\binom{4}{5}\right)$

$= \frac{-1}{49-16}\begin{bmatrix} 7 & -4 \\ -4 & 7 \end{bmatrix}\binom{4}{5}$

$= \frac{-1}{33}\binom{28-20}{-16+35} = \frac{-1}{33}\binom{8}{19}$

(2) Decent dir. if $(\nabla f + \kappa\nabla\phi)'\Delta z_{nt} < 0$

$\left(Qx + c + \frac{\kappa 2}{x'x - \alpha} x\right)'\left(\frac{-1}{33}\binom{8}{19}\right) = \left(\binom{1}{1} + \binom{1}{2} + 2\binom{1}{1}\right)'\left(\frac{-1}{33}\binom{8}{19}\right) = \frac{-1}{33}\binom{4}{5}'\binom{8}{19} = \frac{-127}{33} < 0$ ✓

b) Alternating Direction Method of Multipliers

$$\begin{aligned} &\min\ f(x) + g(y) \\ &\text{s.t. } Ax + By = b \end{aligned} \tag{2}$$

i) Give functions $f$, $g$ and matrices $A$, $B$ and $b$ so that problem (2) is equivalent to (1).
*Hint: You may want to use an indicator function.*

ii) Give the three steps of the ADMM algorithm for the functions and data you gave in part i)

$$x^{k+1} = \text{argmin}_x\ \ f(x) + \frac{\rho}{2}\|Ax + By^k - b + \mu^k\|^2$$

$$y^{k+1} = \text{argmin}_y\ \ g(y) + \frac{\rho}{2}\|Ax^{k+1} + By - b + \mu^k\|^2$$

$$\mu^{k+1} = \mu^k + Ax^{k+1} + By^{k+1} - b$$

i) $f(x) = \frac{1}{2}x'Qx + c'x$ $\quad A = I,\ B = -I,\ b = 0$

$g(y) = \begin{cases} 0 & \text{if } y'y \leq \alpha \\ \infty & \text{otherwise} \end{cases}$

ii) $x^{k+1} = \text{argmin}\ \frac{1}{2}x'Qx + c'x + \frac{\rho}{2}\|x - y^k + \mu^k\|^2$

$\nabla = Qx + c + \rho(x - y^k + \mu^k) = 0$

$(Q + \rho I)x = -c - \rho(\mu^k - y^k)$

$x^{k+1} = (Q + \rho I)^{-1}(-c - (\mu^k - y^k))$

$y^{k+1} = \underset{y'y \leq \alpha}{\text{argmin}}\ \|x^{k+1} - y + \mu^k\|^2 = \begin{cases} x^{k+1} + \mu^k & \text{if } \|x^{k+1} + \mu^k\|^2 \leq \alpha \\ \dfrac{x^{k+1} + \mu^k}{\|x^{k+1} + \mu^k\|} \cdot \sqrt{\alpha} & \text{otherwise} \end{cases}$

$\mu^{k+1} = \mu^k + x^{k+1} - y^{k+1}$

**Problem 3.**

/15

Consider the system $x^+ = Ax + Bu$ with the state constraint $x \in X$ and input constraint $u \in U$.

Let $C \subseteq X$ be a control invariant set for this system and consider the following MPC controller.

$$\begin{aligned} \min \ & \sum_{i=0}^{N-1} x_i^T Q x_i + u_i^T R u_i \\ \text{s.t. } & x_1 \in C \\ & u_i \in U \quad i \in \{0, \dots, N-1\} \\ & x_i \in 2 \cdot X \quad i \in \{1, \dots, N\} \\ & x_{i+1} = Ax_i + Bu_i \end{aligned}$$

Is the resulting closed-loop system recursively feasible?

☒ Yes ◯ No

If yes, then prove it. If no, then provide a counter-example.

Let $\{x_0, \dots, x_N, u_0, \dots, u_{N-1}\}$ be a feasible sol$^n$.

At the next time instant, the state will be $x^+ = Ax_0 + Bu_0 = x_1$

Because the sequence above was feasible, we have that $x_1 \in C$.

Therefore, there exists a feasible sequence of infinite length (and therefore one of length $N$) that begins at $x_1$ and remains in the constraint set $X$. Since $X \subset 2X$, this sequence is also feasible for the above problem. ∎

**Problem 4.**

/15

Consider the linear system

$$x^+ = \begin{bmatrix} 0.5 & 0 \\ 4 & 0.8 \end{bmatrix} x + \begin{bmatrix} 0.3 & 0.2 \\ -0.6 & 0.9 \end{bmatrix} u$$

with constraints on the input $\|u\|_\infty \leq 1$.

a) What is the maximum control invariant set for this system? Justify your answer.

$\mathbb{R}^2$

Take $u=0$. The max invar set is then $\mathbb{R}^2$, and the input constraints are met everywhere in $\mathbb{R}^2$.

b) Consider the following standard MPC optimization problem, and let $\pi(x)$ be the resulting receding-horizon control law.

$$\begin{aligned} \min \; & \sum_{i=0}^{N-1} x_i^T Q x_i + u_i^T R u_i + x_N^T Q_f x_N \\ \text{s.t. } & x_{i+1} = A x_i + B u_i \\ & u_i \in U \quad i \in \{0, \ldots, N-1\} \\ & x_N \in X_f \\ & x_0 = x \end{aligned}$$

Describe how to choose a terminal control law, $K_f$, terminal weight $Q_f$ and terminal set $X_f$ so that the closed-loop system $x^+ = Ax + B\pi(x)$ has a maximal invariant set equal to that given in Part a) and is asymptotically stable.

We require that there are no state constraints $\Leftrightarrow X_f = \mathbb{R}^2$

Satisfy our standard conditions:

1) $Q, R > 0$

2) Take $K_f(x) = 0$. Then $x^+ = Ax \in \mathbb{R}^2 \;\; \forall x \in \mathbb{R}^2$

$X_f = \mathbb{R}^2 \subseteq \mathbb{R}^2$, $K_f(x) = 0 \in U$ for all $x \in X_f$

3) Define $Q_f$ to satisfy the Lyap condition; $V_f(x) = x' Q_f x$

$V_f(x^+) - V_f(x) \leq -l(x, K_f(x)) \quad \forall x \in X_f$

$x' A' Q_f A x - x' Q_f x \leq -x' Q x$

$\Leftrightarrow A' Q_f A - Q_f \leq -Q$

**Problem 5.**

/15

Consider the uncertain system $x^+ = \frac{1}{2}x + w$ under the state constraint $-10 \le x \le 10$ and subject to a disturbance bounded to lie in the set $|w| \le 1$.

a) Give an algorithm to compute the minimum robust invariant set

$\Omega_0 \leftarrow \{0\}$

loop

$\Omega_{i+1} \leftarrow \Omega_i \oplus A^i W$

if $\Omega_{i+1} = \Omega_i$ then

return $F_\infty = \Omega_i$

end if

end loop

b) Compute the minimum robust invariant set
*Hint:* $[a, b] \oplus [c, d] = [a + c, b + d]$

$\Omega_0 = \{0\}$

$\Omega_1 = \{0\} \oplus \frac{1}{2^0} \cdot [-1 \;\; 1] = [-1 \;\; 1]$

$\Omega_2 = [-1 \;\; 1] \oplus \frac{1}{2^1} \cdot [-1 \;\; 1] = [-1 - \frac{1}{2} \;\; 1 + \frac{1}{2}]$

$\Omega_3 = [-1 - \frac{1}{2} \;\; 1 + \frac{1}{2}] \oplus \frac{1}{2^2} \cdot [-1 \;\; 1] = [-1 - \frac{1}{2} - \frac{1}{4} \;\; 1 + \frac{1}{2} + \frac{1}{4}]$

$\vdots$

$\Omega_\infty = \sum_{i=0}^{\infty} \frac{1}{2^i} [-1 \;\; 1] = \frac{2}{2-1} \cdot [-1 \;\; 1] = [-2 \;\; 2]$

c) Give an algorithm to compute the maximum robust invariant set

$\Omega_0 \leftarrow X$
loop
$\Omega_{i+1} \leftarrow \Omega_i \cap \text{pre}^W(\Omega_i)$
if $\Omega_{i+1} = \Omega_i$ then
return $O_\infty = \Omega_i$
endif
end loop

d) Compute the maximum robust invariant set

$\Omega_0 = [-10 \ \ 10]$

$$\text{pre}^W(\Omega_0) = \{x \mid -10 \le \tfrac{1}{2}x + w \le 10 \ , \forall w, -1 \le w \le 1\}$$
$$= \{x \mid -20 - 2w \le x \le 20 + 2w \ , \forall w \in [-1 \ 1]\}$$
$$= \{x \mid -18 \le x \le 18\}$$

$\Omega_1 = [-10 \ \ 10] \cap [-18 \ \ 18]$
$= [-10 \ \ 10]$
$= O_\infty$

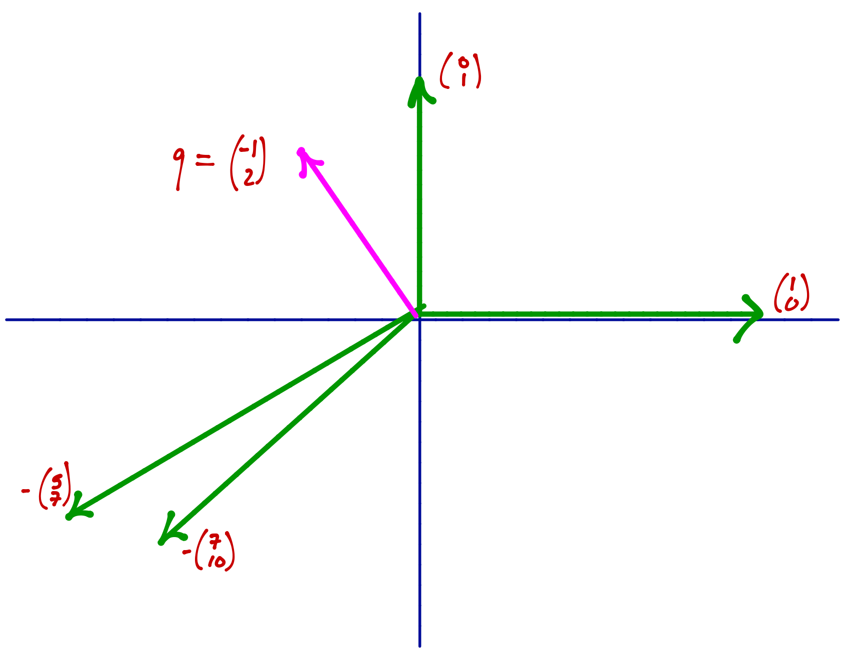
**Problem 6.**

/15

Consider the following linear complementarity problem:

$$w - \begin{bmatrix} 5 & 7 \\ 7 & 10 \end{bmatrix} z = q \quad w^T z = 0 \quad w, z \geq 0$$

a) What is the solution to this LCP for $q = \bar{q} = \begin{bmatrix} -1 & 2 \end{bmatrix}^T$?




Basis is: $w_1 = 0, w_2 \geq 0, z_1 \geq 0, z_2 = 0$

$$\begin{bmatrix} -5 & 0 \\ -7 & 1 \end{bmatrix} \begin{pmatrix} z_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} z_1 \\ w_2 \end{pmatrix} = \begin{bmatrix} -5 & 0 \\ -7 & 1 \end{bmatrix}^{-1} \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \frac{-1}{5} \begin{bmatrix} 1 & 0 \\ 7 & -5 \end{bmatrix} \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \frac{-1}{5} \begin{pmatrix} -1 \\ -17 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 1 \\ 17 \end{pmatrix} > 0 \checkmark$$

Check sol$^n$: $\begin{pmatrix} 0 \\ 1 \end{pmatrix} \frac{17}{5} - \begin{pmatrix} 5 \\ 7 \end{pmatrix} \frac{1}{5} = \frac{1}{5} \begin{pmatrix} -5 \\ 10 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix} \checkmark$

b) Find a matrix $T$ and a vector $t$ such that $\begin{bmatrix} w \\ z \end{bmatrix} = Tq + t$ is the solution to the LCP in a neighbourhood of $\bar{q}$

$$\begin{pmatrix} w \\ z \end{pmatrix} = \underbrace{\frac{-1}{5}\begin{bmatrix} 0 & 0 \\ 7 & -5 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}}_{T} q + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{t}$$

c) Give the neighbourhood $P$ in which the affine function you found in the last part is the solution to the LCP.

$$P = \{ q \,/\, Tq + t \geq 0 \}$$