Model Predictive Control : Exercise 6 - Solutions

Prob 1 | Solving Explicit MPC using parametric LCPs

Consider the discrete-time linear time-invariant system defined by the dynamics

$$x^+ = 2x + u - 1$$

with constraints

$$U = \{u \mid 0 \leq u \leq 2\}$$

We formulate the following MPC problem with horizon $N = 1$:

$$\begin{aligned} f(x) = \min\ & x^2 + u^2 + (x^+)^2 \\ \text{s.t.}\ & x^+ = 2x + u - 1 \\ & 0 \leq u \leq 2 \end{aligned}$$

Your goal is to calculate the explicit solution $f^\star(x)$ of the parametric program and the corresponding explicit control policy $u^\star(x)$.

Tasks:

- To simplify the problem, eliminate the decision variable x^+.

> After elimination, we get the following problem:
>
> $$\begin{aligned} f(x) = \min\ & 5x^2 - 4x + 4xu + 2u^2 - 2u + 1 \\ & u \geq 0, \quad 2 - u \geq 0 \end{aligned}$$

- Write down the Lagrangian function $\mathcal{L}(x, u, \lambda, \nu)$ where λ corresponds to the constraint $0 \leq u$ and ν corresponds to the constraint $u \leq 2$.

> $$\mathcal{L}(x, u, \lambda, \nu) = 5x^2 - 4x + 4xu + 2u^2 - 2u + 1 - \lambda u - \nu(2 - u)$$

- Write down the KKT conditions (stationarity, primal/dual feasibility, complementarity).

The KKT conditions are given by:

Stationarity
$\lfloor \nabla_u \mathcal{L}(x, u, \lambda, \nu) = 4u + 4x - \lambda + \nu - 2 = 0,$

Primal feasibility
$\lfloor s = 2 - u, \quad u, s \geq 0,$

Dual feasibility
$\lfloor \lambda, \nu \geq 0,$

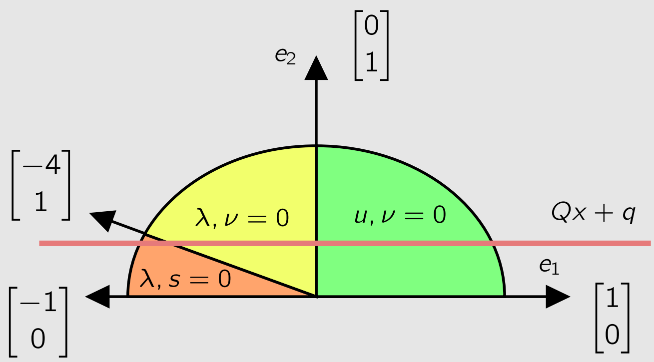
Complementarity
$\lfloor \lambda u = 0, \quad \nu s = 0,$

- Give matrices M, Q and vector q such that the optimal solution of the problem is a linear transform of the solution $y(x)$ to the following parametric LCP:

$$w - My = Qx + q \qquad\qquad w, y \geq 0 \qquad\qquad w^T y = 0$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} \lambda \\ s \end{bmatrix}}_{w} - \underbrace{\begin{bmatrix} 4 & 1 \\ -1 & 0 \end{bmatrix}}_{M} \underbrace{\begin{bmatrix} u \\ \nu \end{bmatrix}}_{y} = \underbrace{\begin{bmatrix} 4 \\ 0 \end{bmatrix}}_{Q} x + \underbrace{\begin{bmatrix} -2 \\ 2 \end{bmatrix}}_{q}, \quad \begin{bmatrix} \lambda \\ s \end{bmatrix}^T \begin{bmatrix} u \\ \nu \end{bmatrix} = 0$$

- Draw the complementarity cones of the pLCP.



Note that there exists no cone between $\begin{bmatrix} -1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

- Compute the optimal value function $f^\star(x)$ and its corresponding control policy $u^\star(x)$.

We are considering 3 different cases, where the cones are intersected by $Qx + q$:

  - $\lambda = 0, s = 0$

$$\begin{bmatrix} u \\ \nu \end{bmatrix} = -\begin{bmatrix} 4 & 1 \\ -1 & 0 \end{bmatrix}^{-1} \left(\begin{bmatrix} 4 \\ 0 \end{bmatrix} x + \begin{bmatrix} -2 \\ 2 \end{bmatrix} \right) = -\begin{bmatrix} 0 & -1 \\ 1 & 4 \end{bmatrix} \left(\begin{bmatrix} 4 \\ 0 \end{bmatrix} x + \begin{bmatrix} -2 \\ 2 \end{bmatrix} \right)$$
$$= \begin{bmatrix} 0 \\ -4 \end{bmatrix} x + \begin{bmatrix} 2 \\ -6 \end{bmatrix} \geq 0 \implies x \leq -1.5$$

$$u^\star(x) = 2, \quad f^\star(x) = 5x^2 + 4x + 5$$

- $\lambda = 0, \nu = 0$

$$\begin{bmatrix} u \\ s \end{bmatrix} = \begin{bmatrix} -4 & 0 \\ 1 & 1 \end{bmatrix}^{-1} \left(\begin{bmatrix} 4 \\ 0 \end{bmatrix} x + \begin{bmatrix} -2 \\ 2 \end{bmatrix} \right) = -\frac{1}{4} \begin{bmatrix} 1 & 0 \\ -1 & -4 \end{bmatrix} \left(\begin{bmatrix} 4 \\ 0 \end{bmatrix} x + \begin{bmatrix} -2 \\ 2 \end{bmatrix} \right)$$
$$= \begin{bmatrix} -1 \\ 1 \end{bmatrix} x + \begin{bmatrix} 0.5 \\ 1.5 \end{bmatrix} \geq 0 \implies -1.5 \leq x \leq 0.5$$

$$u^\star(x) = 0.5 - x, \quad f^\star(x) = 3x^2 - 2x + 0.5$$

- $u = 0, \nu = 0$

$$\begin{bmatrix} \lambda \\ s \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix} x + \begin{bmatrix} -2 \\ 2 \end{bmatrix} \geq 0 \implies 0.5 \leq x$$

$$u^\star(x) = 0, \quad f^\star(x) = 5x^2 - 4x + 1$$

- $u = 0, \nu = 0$

$$\begin{bmatrix} \lambda \\ s \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix} x + \begin{bmatrix} -2 \\ 2 \end{bmatrix} \geq 0 \implies 0.5 \leq x$$

$$u^\star(x) = 0, \quad f^\star(x) = 5x^2 - 4x + 1$$

Hence, this gives us the solutions:

$$u^\star(x) = \begin{cases} 0, & 0.5 \leq x \\ 0.5 - x, & -1.5 \leq x \leq 0.5 \\ 2, & x \leq -1.5 \end{cases}$$

$$f^\star(x) = \begin{cases} 5x^2 + 4x + 5, & 0.5 \le x \\ 3x^2 - 2x + 0.5, & -1.5 \le x \le 0.5 \\ 5x^2 - 4x + 1, & x \le -1.5 \end{cases}$$

- Use Matlab to estimate $u^\star(x)$ and $f^\star(x)$ by solving the optimization problem for a number of different values of x and compare this to your parametric solution.

Prob 2 | Implement explicit MPC using MPT3

We revisit the MPC problem from exercise 4, where we considered the discrete-time linear time-invariant system defined by

$$x^+ = \begin{bmatrix} 0.9752 & 1.4544 \\ -0.0327 & 0.9315 \end{bmatrix} x + \begin{bmatrix} 0.0248 \\ 0.0327 \end{bmatrix} u$$

with constraints

$$X = \{x \mid |x_1| \le 5, |x_2| \le 0.2\} \qquad\qquad U = \{u \mid |u| \le 1.75\}$$

This is a second-order system with a natural frequency of $0.15r/s$, a damping ratio of $\zeta = 0.1$ which has been discretized at $1.5r/s$. The first state is the position, and the second is velocity.

Your goal is to implement an explicit MPC controller for this system with a horizon of $N = 10$ and a stage cost given by $l(x, u) := 10x^T x + u^T u$ using the MPT3 toolbox.

Tasks:

- Define your MPC problem using MPT3. You can proceed as follows:
  - Define the system `sys = LTISystem('A',A,'B',B)`
  - Define the constraints on the signals by setting the values `sys.x.max = ...`, `sys.x.min = ...`, etc
  - Define the stage costs by setting the penalty terms for x and u, e.g., `sys.x.penalty = QuadFunction(Q)`
  - Extract desired sets and weights with `sys.LQRGain`, `sys.LQRPenalty.weight` and `sys.LQRSet`,
  - Set the terminal cost and terminal set with `sys.x.with('terminalPenalty')`, `sys.x.terminalPenalty = QuadFunction(Qf)` and `sys.x.with('terminalSet')`, `sys.x.terminalSet = Xf`,
  - Define the MPC controller with `controller = MPCController(sys, N)`.
- Generate the explicit MPC with `empc = controller.toExplicit()`.
- Plot the generated solution, including regions, with `empc.feedback.fplot()`.
- Simulate the closed-loop system starting from the state $x = \begin{bmatrix} 3 & 0 \end{bmatrix}^T$.
  Confirm that your constraints are met. Reuse the simulation code from exercise 4. You can evaluate the explicit controller with `empc.evaluate(x)`.

Prob 3 | Compare explicit MPC with YALMIP implementation

We now compare the explicit MPC with the YALMIP implementation from exercise 4.

Tasks:

- If (for some reason) you skipped exercise 4, implement the controller using YALMIP.
- Plot the position, velocity, and input of the system using the YALMIP controller. Confirm that your solution is the same as for the explicit MPC case.
- Compare the solve times of the explicit MPC against the YALMIP implementation. What do you notice, is it as expected?