SOLUTIONS

Problem 1.

/21

When there are multiple choices in the following, select all statements that are true.

1. Consider the SISO system $x_{k+1} = Ax_k + Bu_k$, $x_k \in \mathbb{R}^2$.
   - [ ] If all modes of the system go to zero as $k \to +\infty$, then the system is stable but not asymptotically stable.
   - [x] Assume that $\hat{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\hat{u} = 1$ is an unstable equilibrium. Then, also $\bar{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\bar{u} = 0$ is unstable.
   - [ ] If all free states are bounded, then the system is asymptotically stable.

2. Consider the SISO system $x_{k+1} = Ax_k + Bu_k$, $x_k \in \mathbb{R}^n$.
   - [ ] For every constant input $u_k = \bar{u} \in \mathbb{R}$ there is an equilibrium state.
   - [x] If $x(k) = \phi(k, 0, x_0, 0)$ and $\tilde{x}(k) = \phi(k, 0, 3x_0, u)$, then $\phi(k, 0, 2x_0, u) = \tilde{x}(k) - x(k)$.
   - [x] If $A = 0$ the system is controllable.

3. Let Σ_1 and Σ_2 be two equivalent LTI systems.
   - [x] If Σ_1 is stable, then also Σ_2 is stable.
   - [x] If Σ_1 is asymptotically stable, then Σ_2 is exponentially stable.
   - [x] If Σ_1 is reachable, then Σ_2 is controllable.

4. Consider the system $x_{k+1} = Ax_k + Bu_k$, $y_k = Cx_k$, $x_k \in \mathbb{R}^n$, $u_k \in \mathbb{R}^m$, $y_k \in \mathbb{R}^p$.
   - [ ] A state $\bar{x} \in \mathbb{R}^n$ is unobservable if there is an input such that $\phi(k, 0, \bar{x}, u) = 0$, $\forall k \geq 0$.
   - [x] If the states $\bar{x}$ and $\tilde{x}$ are unobservable, then also $\bar{x} + \tilde{x}$ is unobservable.
   - [x] Assume that the observability matrix has full rank and $u_k = 0$, $\forall k \geq 0$. Then $y_k = 0$, $k = 0, 1, \ldots, n$ implies that $x(0) = 0$.

5. Consider the continuous-time system $\dot{x} = Ax$, $x \in \mathbb{R}^n$.

   - [ ] If $\lambda < 0$ is an eigenvalue of A, then, for any sampling period $T > 0$, the discrete-time system obtained through exact discretization has an eigenvalue $\tilde{\lambda} < 0$.
   - [ ] If all eigenvalues of A are distinct, then also the eigenvalues of $\hat{A} = e^{AT}$ have the same property, $\forall T > 0$.
   - [x] If all eigenvalues of A have real part strictly less than -2, there is $T > 0$ such that the discrete-time system obtained through forward Euler discretization is asymptotically stable.

6. Let $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \sim N\left(\begin{bmatrix} -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\right)$ and $y = x_1 - 1$. Provide the probability density of $z = \begin{bmatrix} x \\ y \end{bmatrix}$.

   z is Gaussian with mean $\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$ and Variance $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

7. Consider the first-order system $x_{k+1} = x_k + 2u_k$ and the cost $J = x_2^2 + \sum_{k=0}^{1} x_k^2 + 2u_k^2$. Compute the gain $K_1 \in \mathbb{R}$ defining the optimal control $u_1 = -K_1 x_1$.

   $K_1 = \frac{1}{3}$

Problem 2.

/24

Consider the system $x_{k+1} = Ax_k + Bu_k$ where

$$A = \begin{bmatrix} 1-a & 0 & 0 \\ c & 1-b & b \\ a & c & a+c \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

i) For $a = 0$, $b = 0$, and $c = 1$ study the stability of the system.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$\lambda_1 = 1$ with algebraic multiplicity $n_1 = 3$

↳ the system can be either stable or unstable

Computation of the geometric multiplicity of λ_1

$$V_1 = \{ v : (A - I)v = 0 \} = \left\{ v : \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 0 \right\} =$$

$$= \left\{ v = \begin{bmatrix} 0 \\ 0 \\ \alpha \end{bmatrix}, \alpha \in \mathbb{R} \right\} \rightarrow \nu_1 = \dim(V_1) = 1$$

Since $n_1 < \nu_1$ the system is unstable

ii) For $a = 0$, $c = 0$ and $b \in \mathbb{R}$ the system has an eigenvalue $\lambda = 1$. Find the values of b such that λ is reachable.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1-b & b \\ 0 & 0 & 0 \end{bmatrix}$$

PBH test : λ is reachable if $\text{rank}\left(\begin{bmatrix} \lambda I - A & B \end{bmatrix}\right) = 3$

$$\text{rank}\left(\begin{bmatrix} 0 & \underbrace{\begin{matrix} 0 & 0 & 1 \\ +b & -b & 0 \\ 0 & 1 & -1 \end{matrix}}_{L} \end{bmatrix}\right) = \text{rank}(L). \text{ Since } \det(L) = b,$$

λ is reachable for $b \neq 0$.

iii) For $a = 0$, $b = 0$, and $c = 0$ compute a change of state coordinates $T^{-1}\hat{x} = x$ such that the system $\hat{x}_{k+1} = \hat{A}\hat{x}_k + \hat{B}\hat{u}_k$ is in reachability form. Give the block structure of $\hat{A}$ and $\hat{B}$, and specify the dimension of the zero blocks (do not compute the entries of other blocks).

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow AB = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad A^2B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$M_r = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \rightarrow \text{rank}(M_r) = 2$$

$$\begin{matrix} \downarrow & \downarrow \\ v_1 & v_2 \end{matrix}$$

- Change of coordinates $T^{-1} = [v_1, v_2, v_3]$ where v_3 makes T invertible (e.g. $v_3 = [0 \ 1 \ 0]^T$)
- Block structure

$$\hat{A} = \begin{bmatrix} \hat{A}_a & \hat{A}_{ab} \\ \underset{1\times 2}{0} & \hat{A}_b \end{bmatrix} \qquad \hat{B} = \begin{bmatrix} \hat{B}_a \\ \underset{\text{scalar}}{0} \end{bmatrix}$$

Problem 3.

/20

Consider the system

$$x_1^+ = 0.5x_1 + x_2 + u$$
$$x_2^+ = x_2$$
$$y = x_1 + x_2$$

Design a reduced-order observer with zero eigenvalues.

$$A = \begin{bmatrix} 0.5 & 1 \\ 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

Observability matrix $M_O = \begin{bmatrix} 1 & 0.5 \\ 1 & 2 \end{bmatrix} \rightarrow$ full rank

Transformation in the state-space $\bar{x} = Tx$

$T = \begin{bmatrix} C \\ T_1 \end{bmatrix}$ such that $\det(T) \neq 0$. This is OK

for $T_1 = \begin{bmatrix} 0 & 1 \end{bmatrix} \rightarrow T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. $T^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$

Equivalent system $\bar{A} = TAT^{-1}$ $\bar{B} = TB$ $\bar{C} = CT^{-1}$

$$\bar{A} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0.5 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{3}{2} \\ 0 & 1 \end{bmatrix}$$

$$\bar{B} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$\bar{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}$ (by construction)

$$\bar{x}^{+} = \begin{bmatrix} \frac{1}{2} & \frac{3}{2} \\ 0 & 1 \end{bmatrix} \bar{x} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

$$y = \bar{x}_1$$

Setting $\bar{x} = \begin{bmatrix} y \\ w \end{bmatrix}$ we have

$$\begin{cases} y^{+} = \frac{1}{2} y + \frac{3}{2} w + u & \longrightarrow \quad \frac{3}{2} w = \underbrace{y^{+} - \frac{1}{2} y - u}_{\text{measured output } \bar{y}(k+1)} \\ w^{+} = w \end{cases}$$

$$\begin{cases} w^{+} = \bar{A}_{22} w & \bar{A}_{22} = 1 \\ \bar{y}(k+1) = \bar{A}_{12} w & \bar{A}_{12} = \frac{3}{2} \end{cases}$$

Full-order observer (with no delay) for w

$$\hat{w}^{+} = \hat{w} - L\left(\bar{y}(k+1) - \frac{3}{2} w\right)$$

Since $(\bar{A}_{22}, \bar{A}_{12}) = (1, \frac{3}{2})$ is observable, then L can be designed such that $|\bar{A}_{22} + L\bar{A}_{12}| =$

$$= \left|1 + L\frac{3}{2}\right| < 1$$

$\hookrightarrow$ For $L = -\frac{2}{3}$ the observer eigenvalue is in the origin

Problem 4.

/20

Consider the system

$$x_1^+ = -x_2$$
$$x_2^+ = -x_1 - \frac{1}{2}x_2 + u + d$$
$$y = x_1$$

where $d(k) \in \mathbb{R}$ is a constant but unknown disturbance. Assume that both states are measured. After verifying the necessary assumptions, design a controller for tracking perfectly a constant reference $y^o(k) \in \mathbb{R}$ after a finite number of steps. For simplicity, do not solve any subproblem requiring eigenvalue assignment: just state the subproblem and discuss if it can be solved.

LTI model

$$x^+ = Ax + Bu + Md \qquad A = \begin{bmatrix} 0 & -1 \\ -1 & -\frac{1}{2} \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad M = B$$

$$y = Cx + Nd \qquad C = \begin{bmatrix} 1 & 0 \end{bmatrix} \qquad N = 0$$

We want do design a controller with integral action

- A first necessary condition is

$$\det(\Sigma) \neq 0 \qquad \Sigma = \begin{bmatrix} A - I & B \\ C & 0 \end{bmatrix}$$

$$\text{check}: \Sigma = \begin{bmatrix} -1 & -1 & 0 \\ -1 & -\frac{3}{2} & 1 \\ 1 & 0 & 0 \end{bmatrix} \rightarrow \text{rank } 3 \quad \text{OK}$$

- The second necessary condition is that (A,B) is reachable

$$M_r = [B, AB] = \begin{bmatrix} 0 & -1 \\ 1 & -\frac{1}{2} \end{bmatrix} \rightarrow \text{full rank.}$$

Design of the controller

- Add an integrator

$$\begin{cases} v^+ = v + (y^o - y) = v + y^o - Cx - Nd = v + y^o - Cx \\ w = v \end{cases}$$

- Build the extended system with state $\eta^T = [x_1, x_2\ v]$ and output w

$$\eta^+ = \underbrace{\begin{bmatrix} 0 & -1 & 0 \\ -1 & -\frac{1}{2} & 0 \\ -1 & 0 & 1 \end{bmatrix}}_{\bar{A}} \eta + \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}_{\bar{B}} u + \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}_{\bar{M}} d + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} y_o$$

$$w = \underbrace{[0 \quad 0 \quad 1]}_{\bar{C}} \eta + \underbrace{0}_{\bar{D}} d$$

- We must find $u = K\eta$ such that $(\bar{A} + \bar{B}K)$ has all eigenvalues equal to zero → deadbeat behavior → the error $y^o - y$ will go to zero in 3 steps
- The problem can be solved because (A, B) reachable and Σ full rank imply that $(\bar{A}, \bar{B})$ is reachable.

Problem 5.

/15

Consider the first-order system

$$
\begin{aligned}
x_{k+1} &= 2x_k - u_k + w_k & \quad w &\sim \mathrm{WGN}(0,1) \\
y_k &= x_k + v_k & \quad \cancel{w}\, v &\sim \mathrm{WGN}(0,2) \\
x_0 &\sim N(2, 0.1)
\end{aligned}
$$

1. Assuming that the noise terms are zero, compute the LQ regulator minimizing the cost

$$J = \sum_{k=0}^{+\infty} Q x_k^2 + R u_k^2, \quad Q = 2, \; R = 1.$$

and the corresponding closed-loop eigenvalue.

System matrices : $A = 2 \quad B = -1 \quad C = 1$

Computation of the LQ regulator

DARE

$$P = A^T P A + Q - A^T P B \left[B^T P B + R \right]^{-1} B^T P A$$

$$P = 4P + 2 - \frac{4P^2}{P+1}$$

$$(P+1)\, P = \cancel{4P^2} + 2P + 4P + 2 - \cancel{4P^2} \rightarrow P^2 - 5P - 2 = 0$$

$$P = \frac{5 \pm \sqrt{25+8}}{2} = \frac{5 \pm \sqrt{33}}{2} = 5.3723$$

$\uparrow$ $P > 0$

The LQ controller is

$$u = -K x_k$$

$$K = (B^T P B + R)^{-1} B^T P A = -\frac{P}{P+1}\, 2 = -1.6861$$

Closed-loop eigenvalue : $A - BK = 0.3139$

2. Compute the steady-state Kalman predictor.

DARE $(W=1,\ V=2)$

$$\Sigma = A\Sigma A^T + W - A\Sigma C^T[C\Sigma C^T+V]^{-1}C\Sigma A$$

$$\Sigma = 4\Sigma + 1 - \frac{4\Sigma^2}{\Sigma+2} \qquad (\Sigma+2)\Sigma = \cancel{4\Sigma^2} + \Sigma + 8\Sigma + 2 - \cancel{4\Sigma^2}$$

$$\Sigma^2 - 7\Sigma - 2 = 0$$

$$\Sigma = \frac{7 \pm \sqrt{49+8}}{2} \underset{\substack{\uparrow \\ \Sigma \geq 0}}{=} \frac{7+\sqrt{57}}{2} = 7.2749$$

Kalman gain $\quad L = A^T\Sigma C^T(C\Sigma C^T+V)^{-1} = \dfrac{2\Sigma}{\Sigma+2} = 1.5687$

Predictor : $\hat{x}_{k+1|k} = A\hat{x}_{k|k-1} + Bu_k + L(y_k - C\hat{x}_{k|k})$

$$= \underbrace{(A-LC)}_{0.4313}\hat{x}_{k|k-1} + Bu_k + Ly_k$$

3. Compute the overall output feedback LQG control law minimizing the cost

$$J = \lim_{N\to+\infty}\frac{1}{N}\mathbb{E}\left[\sum_{k=0}^{N-1} Qx_k^2 + Ru_k^2\right], \quad Q=2,\ R=1.$$

and provide the eigenvalues of the closed-loop system.

The LQG law is $u_k = -K\hat{x}_{k|k-1}$, together with the Kalman predictor. From the separation principle, the closed-loop eigenvalues are 0.3139 and 0.4313

4. Assume, as in point 1, that the noise terms are zero. Explain how to modify the optimal control problem in point 1 for guaranteeing that the closed-loop eigenvalue λ verifies $|\lambda| \leq \frac{1}{4}$.

As seen in the lecture, by defining $\hat{A} = \alpha A$, $\hat{B} = \alpha B$ and by minimizing the cost for the system $\hat{x}^+ = \hat{A}x + \hat{B}u$, one obtains a controller gain K_α such that $|A - BK_\alpha| \leq \frac{1}{\alpha}$.

It is therefore enough to set $\alpha = 4$.

5. Assume that $y_0 = 2$, $y_1 = 3$, and $u_0 = u_1 = 0$. Compute $E[x_2|y_0, y_1]$.

The problem can be solved using the time-varying KF. For $k = 0$

$$\hat{x}_{0|-1} = E[x_0] = 2 \qquad \Sigma_{0|-1} = Var[x_0] = \frac{1}{10}$$

$$\hat{x}_{0|0} = \hat{x}_{0|-1} + \Sigma_{0|-1} C^T (C^T \Sigma_{0|-1} C + V)^{-1} \underbrace{(y_0 - C\hat{x}_{0|-1})}_{=0} =$$

$$= 2$$

$$\Sigma_{0|0} = \Sigma_{0|-1} - \Sigma_{0|-1} C^T (C \Sigma_{0|-1} C^T + V)^{-1} C \Sigma_{0|-1} =$$

$$= \frac{1}{10} - \frac{1}{10} \left(\frac{1}{10} + 2\right)^{-1} \frac{1}{10} = \frac{1}{10} - \frac{1}{210} = 0.0952$$

$$\hat{x}_{1|0} = A\,\hat{x}_{0|0} = 4$$

$$\Sigma_{1|0} = A\,\Sigma_{0|0}\,A^T + W = 4 \cdot 0.0952 + 1 = 1.381$$

At $k = 1$

$$\hat{x}_{1|1} = \hat{x}_{1|0} + \Sigma_{1|0}\,C^T\left(C\,\Sigma_{1|0}\,C^T + V\right)^{-1}\left(y_1 - C\,\hat{x}_{1|0}\right) =$$

$$= 4 + 1.381\,(1.381 + 2)^{-1}\,(3 - 4) = 3.5915$$

$$\hat{x}_{2|1} = A\,\hat{x}_{1|1} = 2 \cdot 3.5915 = 7.1831$$