Problem 1.

/21

When there are multiple choices in the following, select all statements that are true.

1. Consider the SISO system $x_{k+1} = Ax_k + Bu_k$, $x_k \in \mathbb{R}^2$.
   - ◯ If all modes of the system go to zero as $k \to +\infty$, then the system is stable but not asymptotically stable.
   - ◯ Assume that $\hat{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\hat{u} = 1$ is an unstable equilibrium. Then, also $\bar{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\bar{u} = 0$ is unstable.
   - ◯ If all free states are bounded, then the system is asymptotically stable.

2. Consider the SISO system $x_{k+1} = Ax_k + Bu_k$, $x_k \in \mathbb{R}^n$.
   - ◯ For every constant input $u_k = \bar{u} \in \mathbb{R}$ there is an equilibrium state.
   - ◯ If $x(k) = \phi(k, 0, x_0, 0)$ and $\tilde{x}(k) = \phi(k, 0, 3x_0, u)$, then $\phi(k, 0, 2x_0, u) = \tilde{x}(k) - x(k)$.
   - ◯ If $A = 0$ the system is controllable.

3. Let Σ_1 and Σ_2 be two equivalent LTI systems.
   - ◯ If Σ_1 is stable, then also Σ_2 is stable.
   - ◯ If Σ_1 is asymptotically stable, then Σ_2 is exponentially stable.
   - ◯ If Σ_1 is reachable, then Σ_2 is controllable.

4. Consider the system $x_{k+1} = Ax_k + Bu_k$, $y_k = Cx_k$, $x_k \in \mathbb{R}^n$, $u_k \in \mathbb{R}^m$, $y_k \in \mathbb{R}^p$.
   - ◯ A state $\bar{x} \in \mathbb{R}^n$ is unobservable if there is an input such that $\phi(k, 0, \bar{x}, u) = 0$, $\forall k \geq 0$.
   - ◯ If the states $\bar{x}$ and $\tilde{x}$ are unobservable, then also $\bar{x} + \tilde{x}$ is unobservable.
   - ◯ Assume that the observability matrix has full rank and $u_k = 0$, $\forall k \geq 0$. Then $y_k = 0$, $k = 0, 1, \dots, n$ implies that $x(0) = 0$.

5. Consider the continuous-time system $\dot{x} = Ax$, $x \in \mathbb{R}^n$.

   ◯ If $\lambda < 0$ is an eigenvalue of A, then, for any sampling period $T > 0$, the discrete-time system obtained through exact discretization has an eigenvalue $\tilde{\lambda} < 0$.

   ◯ If all eigenvalues of A are distinct, then also the eigenvalues of $\hat{A} = e^{AT}$ have the same property, $\forall T > 0$.

   ◯ If all eigenvalues of A have real part strictly less than -2, there is $T > 0$ such that the discrete-time system obtained through forward Euler discretization is asymptotically stable.

6. Let $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \sim N\left(\begin{bmatrix} -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\right)$ and $y = x_1 - 1$. Provide the probability density of $z = \begin{bmatrix} x \\ y \end{bmatrix}$.

7. Consider the first-order system $x_{k+1} = x_k + 2u_k$ and the cost $J = x_2^2 + \sum_{k=0}^{1} x_k^2 + 2u_k^2$. Compute the gain $K_1 \in \mathbb{R}$ defining the optimal control $u_1 = -K_1 x_1$.

Problem 2.

/24

Consider the system $x_{k+1} = Ax_k + Bu_k$ where

$$A = \begin{bmatrix} 1-a & 0 & 0 \\ c & 1-b & b \\ a & c & a+c \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

i) For $a = 0$, $b = 0$, and $c = 1$ study the stability of the system.

ii) For $a = 0$, $c = 0$ and $b \in \mathbb{R}$ the system has an eigenvalue $\lambda = 1$. Find the values of b such that λ is reachable.

iii) For $a = 0$, $b = 0$, and $c = 0$ compute a change of state coordinates $T^{-1}\hat{x} = x$ such that the system $\hat{x}_{k+1} = \hat{A}\hat{x}_k + \hat{B}\hat{u}_k$ is in reachability form. Give the block structure of $\hat{A}$ and $\hat{B}$, and specify the dimension of the zero blocks (do not compute the entries of other blocks).

Problem 3. /20

Consider the system

$$
\begin{aligned}
x_1^+ &= 0.5x_1 + x_2 + u \\
x_2^+ &= x_2 \\
y &= x_1 + x_2
\end{aligned}
$$

Design a reduced-order observer with zero eigenvalues.

Problem 4.

/20

Consider the system

$$
\begin{aligned}
x_1^+ &= -x_2 \\
x_2^+ &= -x_1 - \frac{1}{2}x_2 + u + d \\
y &= x_1
\end{aligned}
$$

where $d(k) \in \mathbb{R}$ is a constant but unknown disturbance. Assume that both states are measured. After verifying the necessary assumptions, design a controller for tracking perfectly a constant reference $y^o(k) \in \mathbb{R}$ after a finite number of steps. For simplicity, do not solve any subproblem requiring eigenvalue assignment: just state the subproblem and discuss if it can be solved.

Problem 5.

/15

Consider the first-order system

$$\begin{aligned} x_{k+1} &= 2x_k - u_k + w_k & w &\sim \mathrm{WGN}(0,1) \\ y_k &= x_k + v_k & v &\sim \mathrm{WGN}(0,2) \\ x_0 &\sim N(2, 0.1) \end{aligned}$$

1. Assuming that the noise terms are zero, compute the LQ regulator minimizing the cost

$$J = \sum_{k=0}^{+\infty} Qx_k^2 + Ru_k^2, \quad Q = 2,\ R = 1.$$

and the corresponding closed-loop eigenvalue.

2. Compute the steady-state Kalman predictor.

3. Compute the overall output feedback LQG control law minimizing the cost

$$J = \lim_{N \to +\infty} \frac{1}{N} \mathbb{E} \left[\sum_{k=0}^{N-1} Q x_k^2 + R u_k^2 \right], \quad Q = 2, \ R = 1.$$

and provide the eigenvalues of the closed-loop system.

4. Assume, as in point 1, that the noise terms are zero. Explain how to modify the optimal control problem in point 1 for guaranteeing that the closed-loop eigenvalue λ verifies $|\lambda| \leq \frac{1}{4}$.

5. Assume that $y_0 = 2$, $y_1 = 3$, and $u_0 = u_1 = 0$. Compute $\mathrm{E}[x_2 | y_0, y_1]$.