Multivariable Control (ME-422) - Exercise session 9 SOLUTIONS

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1. Consider the system

$$\begin{cases} x_1^+ = (1-\alpha)x_1 + \beta x_2 - u + \tilde{d} \\ x_2^+ = \alpha x_1 + (1-\beta)x_2 \end{cases}$$
$$y = x_1$$

where $\alpha = 0.5$ and $\beta = 0.5$. Assume that the disturbance is generated by the LTI system

$$x_d^+ = 2x_d$$
$$\tilde{d} = x_d$$

Design a controller based on disturbance estimation for guaranteeing that $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \to 0$.

Solution: We consider the following scheme, where the estimated disturbance is used for compensation.

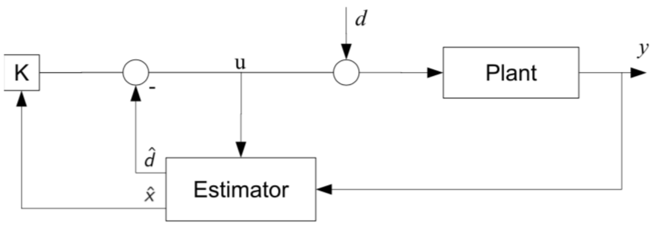

Figure 1: Offset-free tracking based on disturbance estimation

In order to match the above scheme and provide as input to the system $u + d$, we set $d = -\tilde{d}$.

Our model fulfills the assumption that the disturbance acts on the control variable. By matching the notation used in the lectures, we have the plant dynamics as

$$x^+ = Ax + B(u + d)$$
$$y = Cx$$

and the disturbance dynamics as

$$x_d^+ = \phi x_d$$
$$d = Hx_d$$

where the matrices are defined as

$$A = \begin{bmatrix} 1-\alpha & \beta \\ \alpha & 1-\beta \end{bmatrix} \quad B = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$
$$\phi = 2 \quad H = -1$$

The augmented dynamics (plant + disturbance generator) is

$$\underbrace{\begin{bmatrix} x^+ \\ x_d^+ \end{bmatrix}}_{\bar{x}^+} = \underbrace{\begin{bmatrix} A & BH \\ 0 & \phi \end{bmatrix}}_{\bar{A}} \begin{bmatrix} x \\ x_d \end{bmatrix} + \underbrace{\begin{bmatrix} B \\ 0 \end{bmatrix}}_{\bar{B}} u$$

$$y = \underbrace{\begin{bmatrix} C & 0 \end{bmatrix}}_{\bar{C}} \begin{bmatrix} x \\ x_d \end{bmatrix}$$

Let us design an observer for the extended system. For this purpose, we first show that the pair $\left(\bar{A}, \bar{C}\right)$ is observable because

$$M_o = \begin{bmatrix} \bar{C}^T & \bar{A}^T \bar{C}^T & \left(\bar{A}^T\right)^2 \bar{C}^T \end{bmatrix} = \begin{bmatrix} 1 & 0.5 & 0.5 \\ 0 & 0.5 & 0.5 \\ 0 & 1 & 2.5 \end{bmatrix}$$

is full rank.

An observer with eigenvalues at 0.2, 0.3, and 0.4 is given by

$$\begin{aligned} \hat{\bar{x}}^+ &= \bar{A}\hat{\bar{x}} + \bar{B}u - L\left(y - \hat{y}\right) \\ \hat{y} &= \bar{C}\hat{\bar{x}} \end{aligned}$$

where the observer gain L is selected as

$$L = \begin{bmatrix} -2.1 \\ -0.492 \\ -3.264 \end{bmatrix}$$

and computed using the `place` command in MATLAB.

After estimating the states of the extended system and therefore eliminating the disturbance, we now want to design the stabilizing state-feedback controller $u = K\hat{x}$ where $\hat{x}$ collects the first two elements of $\hat{\bar{x}}$ (i.e., the estimation of the plant states x). We first check if the pair (A, B) is reachable. It is, because

$$M_r = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} -1 & -0.5 \\ 0 & -0.5 \end{bmatrix}$$

is full rank.

For the plant, the state-feedback controller gain K assigning the eigenvalues in 0.8 and 0.9 is given by

$$K = \begin{bmatrix} -0.7 & 0.74 \end{bmatrix}$$

and computed, again, using `place` command in MATLAB.

2. **Optimal Feedback Control of a Scalar System**

   The plant to be controlled is the time-invariant scalar system

$$x_{k+1} = ax_k + bu_k \tag{1}$$

   with performance index

$$J = \frac{1}{2}Sx_N^2 + \frac{1}{2}\sum_{k=0}^{N-1}\left(qx_k^2 + ru_k^2\right).$$

   (a) Verify that the closed-loop system obtained by using the FH-LQ controller is given by

$$\begin{aligned} x_{k+1} &= \frac{a}{1 + \left(\frac{b^2}{r}\right)P_{k+1}} x_k \\ P_N &= S \\ P_k &= \frac{a^2 r P_{k+1}}{b^2 P_{k+1} + r} + q \end{aligned} \tag{2}$$

   Derive also the expression of the time-varying gain K_k and of the optimal performance index.

(b) Let $r = 0$, meaning that we do not care how much control is used (i.e., u_k is not weighted in J, so that the optimal solution will make no attempt to keep it small). Find the optimal control law, the optimal cost and relate the intuitive meaning of the cost to the behavior of the closed-loop system.

(c) If we are very concerned not to use too much control energy, we can let $r \to \infty$. In this case, find the closed-form expression of P_k as a function of S.
Hint: For $r \to +\infty$, (2) is an LTI system, for which one can apply the Lagrange formula.

Compute also the gain K_k and the optimal control law. Relate the intuitive meaning of the cost to the closed-loop dynamics.

(d) Write a MATLAB function

```
[K, P] = fhopt(a,b,q,r,S,N)
```

that computes and stores the values of optimal control gain K_k and optimal performance index P_k for $k = 0, \ldots, N-1$.

Simulate the closed-loop system for $a = 1.05$, $b = 0.01$, $q = r = 1$, $x_0 = 10$, $S = 5$, $N = 100$ and plot the sequences P_k, K_k, and x_k. Is the state converging to zero? Why?

Set $S = 500$ and repeat the simulation. What is the difference?

Solution:

(a) In ths scalar case, the Riccati equation is

$$P_N = S$$
$$P_k = a^2 P_{k+1} - \frac{a^2 b^2 P_{k+1}^2}{b^2 P_{k+1} + r} + q \qquad k = N-1, \ldots, 0$$

or, equivalently,

$$P_N = S$$
$$P_k = \frac{a^2 r P_{k+1}}{b^2 P_{k+1} + r} + q \qquad k = N-1, \ldots, 0.$$

The FH-LQ gain is

$$K_k = \frac{abP_{k+1}}{b^2 P_{k+1} + r} = \frac{a/b}{1 + r/b^2 P_{k+1}^{-1}} \tag{3}$$

and the optimal control is

$$u_k = -K_k x_k.$$

The optimal value of the performance index is

$$J^* = \frac{1}{2} P_0 x_0^2.$$

The optimal closed-loop system is

$$x_{k+1} = (a - bK_k)\, x_k = \frac{a}{1 + (b^2/r) P_{k+1}} x_k.$$

(b) In this case, (2) is

$$P_k = q,$$

the feedback gain is $K_k = a/b$, and the optimal control becomes

$$u_k = -\frac{a}{b} x_k.$$

Under the influence of this control, the performance index is

$$J^* = \frac{1}{2} q x_0^2,$$

and the closed-loop system is $x_{k+1} = 0$.

We can understand this as follows. If we have a given value x_k for the state at time k, then a naive approach to minimizing the magnitude of the state vector (which is all we require since $r = 0$) is to solve the state equation in (1) for the u_k required to make x_{k+1} equal to zero, so that $0 = x_{k+1} = a x_k + b u_k$. This yields the control $u = -a/b x_k$.

(c) In this case, (2) becomes

$$P_k = a^2 P_{k+1} + q.$$

The solution to this (Lyapunov) difference equation is

$$P_k = S a^{2(N-k)} + \sum_{i=k}^{N-1} q a^{2(N-i-1)} = S a^{2(N-k)} + \left(\frac{1 - a^{2(N-k)}}{1 - a^2} \right) q.$$

Since P_k is bounded and independent of r, taking the limit of (3) as $r \to \infty$ we have $K_k = 0$, and so the optimal control is $u_k = 0$. The closed-loop system, therefore, is $x_{k+1} = a x_k$.
If we are very concerned about using too much control, the best policy is to use none at all!

(d) See MATLAB file `Ex9.m` for the solution.