Multivariable Control (ME-422) - Exercise session 8 SOLUTIONS

Prof. G. Ferrari Trecate

1. Consider the system

$$\begin{cases} x_1^+ = (1-\alpha)x_1 + \beta x_2 - u + d \\ x_2^+ = \alpha x_1 + (1-\beta)x_2 \end{cases}$$
$$y = x_1$$

where $d(k)$ is a disturbance. Using eigenvalue assignment, we want to design a controller for tracking a constant reference $y^0(k)$ without offset, when the disturbance is constant but unknown.

(a) Check for which parameter values α, $\beta \in \mathbb{R}$ the problem can be solved.

(b) For $\alpha = \frac{1}{2}$ $\beta = \frac{1}{2}$, design the controller.
Hint: First include the integrator dynamics and formulate the dynamics of the augmented system, i.e., system that includes the plant and integrator dynamics. Then, for the augmented system, design a stabilizing state-feedback controller assuming that its states can be measured.

Solution:

(a) LTI model is given by

$$\begin{aligned} x^+ &= Ax + Bu + Md \\ y &= Cx + Nd \end{aligned}$$

where the matrices are

$$A = \begin{bmatrix} 1-\alpha & \beta \\ \alpha & 1-\beta \end{bmatrix} \quad B = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix} \quad M = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad N = 0.$$

Since the disturbance is constant and unknown, we design a controller with integral action for offset-free tracking. One necessary condition to be able to reject disturbance is

$$\det\left(\begin{bmatrix} A-I & B \\ C & 0 \end{bmatrix}\right) \neq 0$$
$$\det\left(\begin{bmatrix} -\alpha & \beta & -1 \\ \alpha & -\beta & 0 \\ 1 & 0 & 0 \end{bmatrix}\right) = (-1)^{3+1}\det\left(\begin{bmatrix} \beta & -1 \\ -\beta & 0 \end{bmatrix}\right) = -\beta \neq 0$$

This condition is satisfied only if $\beta \neq 0$. The second necessary condition is that the pair (A, B) is reachable. This yields

$$M_r = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} -1 & \alpha - 1 \\ 0 & -\alpha \end{bmatrix} \qquad rank(M_r) = 2 \text{ only when } \alpha \neq 0.$$

Therefore, offset-free tracking can only be achieved for parameter values $\alpha \neq 0$ and $\beta \neq 0$. Now, we can specify the integrator dynamics as

$$\begin{aligned} v(k+1) &= v(k) + \big(y^0(k) - y(k)\big) = v(k) + \big(y^0(k) - Cx(k) - Nd(k)\big) \\ w(k) &= v(k) \end{aligned}$$

which yields the extended system with state $\eta = \begin{bmatrix} x_1 & x_2 & v \end{bmatrix}^T$ and dynamics

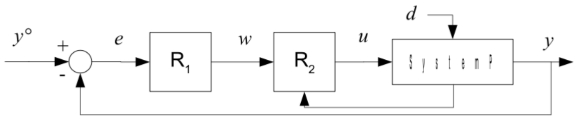
$$\begin{aligned} \eta^+ &= \bar{A}\eta + \bar{B}u + \bar{M}d \\ w &= \bar{C}\eta + \bar{N}d \end{aligned} \tag{1}$$

with matrices defined as

$$\begin{aligned} &\bar{A} = \begin{bmatrix} A & 0 \\ -C & I \end{bmatrix} = \begin{bmatrix} 1-\alpha & \beta & 0 \\ \alpha & 1-\beta & 0 \\ -1 & 0 & 1 \end{bmatrix} \quad \bar{B} = \begin{bmatrix} B \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \quad \bar{M} = \begin{bmatrix} M \\ -N \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\ &\bar{C} = \begin{bmatrix} 0 & I \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \quad \bar{N} = 0 \end{aligned} \tag{2}$$

by taking $y^0 = 0$ without loss of generality, since the system is LTI.

(b) Note the structure of the system with a controller having integrator given in Figure 1b, which was shown in lecture.



Also note that the integrator action given with controller block R_1 in the figure is captured via the dynamics of integration variable $v(k)$, and the dynamics of the augmented system with inputs $(u(k), d(k))$ and output $w(k)$ is given in (1) with matrices in (2) where $\alpha = \beta = \frac{1}{2}$. In order to design the controller R_2, we first check if the pair $(\bar{A}, \bar{B})$ is reachable. This is equivalent to checking the reachability of pair (A, B), as shown in the lecture. Consequently, the value of the parameter $\alpha = \frac{1}{2} \neq 0$ ensures that the system is reachable; therefore, we can design a stabilizing feedback controller for the augmented system.

Now, we will design the controller block R_2 that will stabilize this augmented system. Therefore, select a state-feedback control law $u(k) = K\eta(k)$ with $K \in \mathbb{R}^{1\times 3}$. Replacing this into the augmented system dynamics, we get

$$\begin{aligned} \eta^+ &= \underbrace{\left(\bar{A} + \bar{B}K\right)}_{\bar{A}_{cl}} \eta + \bar{M}d \\ w &= \bar{C}\eta \end{aligned}$$

where the objective is to choose the feedback gain K such that the closed-loop system matrix $\bar{A}_{cl}$ is stable. Then, techniques used in Lecture 6 on designing feedback controllers can be used to design K matrix.

2. Consider the system in the previous problem, but now assume that the disturbance is measured. For $\alpha = \frac{1}{2}$, $\beta = \frac{1}{2}$, after verifying the necessary assumptions, design a feedforward compensator.

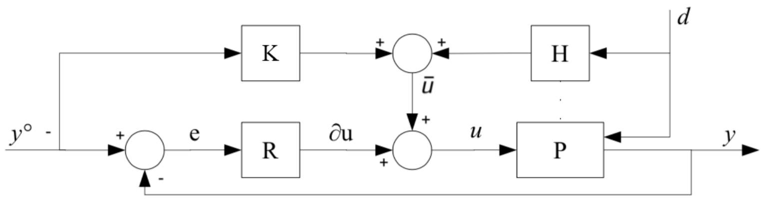


Figure 1: Feedforward Compensator Control Scheme

Figure 1 illustrates the control scheme you will need to design: first design K and H matrices, and then design stabilizing controller R to stabilize the system with inputs $(\partial u, d)$ and output e. To achieve the second objective, first build a Luenberger observer with eigenvalues in 0.5 and 0.6 in order to estimate the states ∂x and then design a state feedback controller $\partial u(k) = K_1 \partial \hat{x}(k)$ with eigenvalues in 0.2 and 0.3.

Solution: The design can be done only if

$$\det(\Sigma) \neq 0 \quad \Sigma = \begin{bmatrix} A - I & B \\ C & 0 \end{bmatrix}.$$

As shown in the previous exercise, this condition is verified for $\alpha = \beta = \frac{1}{2}$. Moreover, for constant y^0 and d, there is a single equilibrium state $\hat{x}$ and corresponding equilibrium input

$$\bar{u} = \underbrace{\begin{bmatrix} 0 & I \end{bmatrix} \Sigma^{-1} \begin{bmatrix} 0 \\ I \end{bmatrix}}_{K} y^0 + \underbrace{\begin{bmatrix} 0 & I \end{bmatrix} \Sigma^{-1} \begin{bmatrix} -M \\ -N \end{bmatrix}}_{H} d$$

One has

$$K = 0 \qquad H = 1.$$

According to the control scheme is given in Figure 1, next step is to design the stabilizing regulator R, which acts on the system

$$\partial P : \begin{cases} \partial x(k+1) = A \partial x(k) + B \partial u(k) \\ e(k) = -C \partial x(k) \end{cases}$$

where $\partial x(k) = x(k) - \bar{x}$, $\partial u = u(k) - \bar{u}$, and $e(k) = y^0 - y(k)$. This can be done by developing an observer coupled with a state-feedback controller. For this design to be possible, we need the pair (A, B) to be reachable and the pair (A, C) to be observable. The first condition is already shown in the solution of the previous exercise. Similarly, the pair (A, C) is observable as

$$M_o = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \qquad rank(M_o) = 2.$$

Thanks to this, we can design a Luenberger observer as

$$\begin{aligned} &\partial \hat{x}(k+1) = A \partial \hat{x}(k) + B \partial u(k) + L\left(e(k) - \hat{e}(k)\right) \\ &\hat{e}(k) = -C \partial \hat{x}(k) \end{aligned}$$

where observer gain L is designed using `place` command of MATLAB so as to set the eigenvalues of $A + LC$ to the desired values 0.5 and 0.6:

$$L = \begin{bmatrix} 0.1 \\ -0.5 \end{bmatrix}$$

Then, one can design the controller K_1 using the same MATLAB command such that the eigenvalues of $A + BK_1$ are set to the desired values 0.2 and 0.3, when using a state feedback controller $\partial u = K_1 \partial \hat{x}$:

$$K_1 = \begin{bmatrix} 0.5 & 0.62 \end{bmatrix}$$

3. **Inverted Pendulum on Cart**

In this exercise, you will design a controller for offset-free tracking of output variable for the cart-pendulum system. This is the same system that you have seen in the previous exercise session. All the files that you will need for the simulation are provided on Moodle.

Download the file *Inverted+pendulum+2018-3+no+solutions.zip* and unzip it. Consider the inverted pendulum described in the file *Inverted+Pendulum+description.pdf* and already used in the

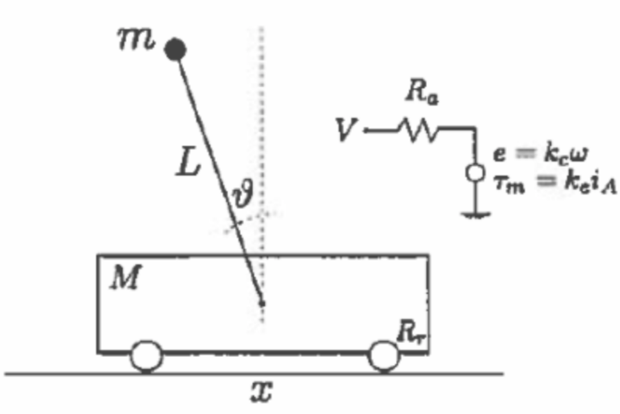


Figure 2: Cart-pendulum system.

previous exercise session. See Figure 2 for the schematic description of the system.

By discretizing, with sampling period $T = 0.1s$, the model obtained through linearization about the equilibrium corresponding to the vertical position, one obtains

$$x^+ = \underbrace{\begin{bmatrix} 1 & 0.0824 & -0.0087 & -0.0003 \\ 0 & 0.6692 & -0.1652 & -0.0087 \\ 0 & 0.0178 & 1.0582 & 0.1019 \\ 0 & 0.3367 & 1.1652 & 1.0582 \end{bmatrix}}_{A_d} x + \underbrace{\begin{bmatrix} 0.0009 \\ 0.0165 \\ -0.0009 \\ -0.0168 \end{bmatrix}}_{B_d} u \tag{3}$$

(a) Set the angular position of the pendulum as the output of the system. Is it possible to track a setpoint $y^0(k) \in \mathbb{R}$, $k = 0, 1, \dots$ by adding integral action? If not, why?

(b) Set the cart position as the output and design an offset-free controller for tracking a reference position $y^0(k) \in \mathbb{R}$, $k = 0, 1, \dots$. Check the conditions guaranteeing that this goal is achievable. To stabilize the extended system, you can assume the whole state is measured, hence avoiding to introduce an observer, and design a controller assigning the closed-loop eigenvalues in e^{-2T}, $e^{-2.5T}$, e^{-3T}, e^{-4T}, and e^{-10T}.

(c) Implement the controller in simulink file *pole_placement_integrator_pendulum_anim_nosol.slx* and verify stability by running simulations for different combinations of the reference inputs. Remember to run the initialization file *pendulum_sys_init.m* prior to running the simulation. **Hint:** In order to implement the integrator dynamics in discrete time, you can use either *Discrete State-Space* block or *Discrete-Time Integrator* block in Simulink. If using the latter option, double-click the block to open the *Block Parameters* menu and make sure to select one of the *Accumulation* methods as the *Integrator Method* option.

Solution:

(a) Taking the angular position of the pendulum as the output corresponds to selecting an output matrix $C_1 = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}$. Then, we see that the system $(A_d, B_d, C_1, 0)$ gives

$$\begin{aligned} \det(\Sigma_1) &= \det\left(\begin{bmatrix} A_d - I & B_d \\ C_1 & 0 \end{bmatrix}\right) \\ &= \det\left(\begin{bmatrix} 0 & -0.0824 & 0.0087 & 0.0003 & 0.0009 \\ 0 & 0.3308 & 0.1652 & 0.0087 & 0.0165 \\ 0 & -0.0178 & -0.0582 & -0.1019 & -0.0009 \\ 0 & -0.3367 & -1.1652 & -0.0582 & -0.0168 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}\right) = 0 \end{aligned}$$

and therefore does not satisfy the necessary condition $det(\Sigma) \neq 0$. Thus, the control design to achieve offset-free tracking is not possible.

Physical Intuition: In steady state, for keeping a pendulum angle $\bar{\theta} \neq 0$, the cart has to drift continuously. But, in this case, the cart position becomes unbounded, which contradicts stability of the closed-loop system and the definition of steady state, i.e., $\hat{x}(k+1) = \hat{x}(k)$.

(b) Taking the position of the cart as the output corresponds to selecting an output matrix $C_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}$. Then, we see that the system $(A_d, B_d, C_2, 0)$ gives

$$\begin{aligned} \det(\Sigma_2) &= \det\left(\begin{bmatrix} A_d - I & B_d \\ C_2 & 0 \end{bmatrix}\right) \\ &= \det\left(\begin{bmatrix} 0 & -0.0824 & 0.0087 & 0.0003 & 0.0009 \\ 0 & 0.3308 & 0.1652 & 0.0087 & 0.0165 \\ 0 & -0.0178 & -0.0582 & -0.1019 & -0.0009 \\ 0 & -0.3367 & -1.1652 & -0.0582 & -0.0168 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}\right) = -1.6330 \times 10^{-4} \neq 0 \end{aligned}$$

and therefore satisfies the necessary condition $det(\Sigma) \neq 0$. Thus, the control design to achieve offset-free tracking is possible.

For achieving offset-free tracking, we will implement a controller with integrator. See the MATLAB file `Ex8.m` for the solution.

(c) See the MATLAB file `Ex8.m` and the Simulink file *pole_placement_integrator_pendulum_anim_sol.slx* for the solution.