Multivariable Control (ME-422) - Exercise session 7

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1. Consider the system

$$\begin{aligned} x_1^+ &= x_1 + x_2 \\ x_2^+ &= x_2 + u \\ y &= x_1 \end{aligned}$$

and design a reduced-order observer, treating the observer gain L as a parameter. Which conditions must L fulfill?

2. **Inverted Pendulum on Cart**
In this exercise, you will design observers to estimate the states of cart-pendulum system that has less outputs than states. This is the same system that you have seen in the previous exercise session. All the files that you will need for the simulation are provided on Moodle.

Download the file *Inverted+pendulum+2018-2+no+solutions.zip* and unzip it. Consider the inverted pendulum described in the file *Inverted+Pendulum+description.pdf* and already used in the previous exercise session. See Figure 1 for the schematic description of the system.

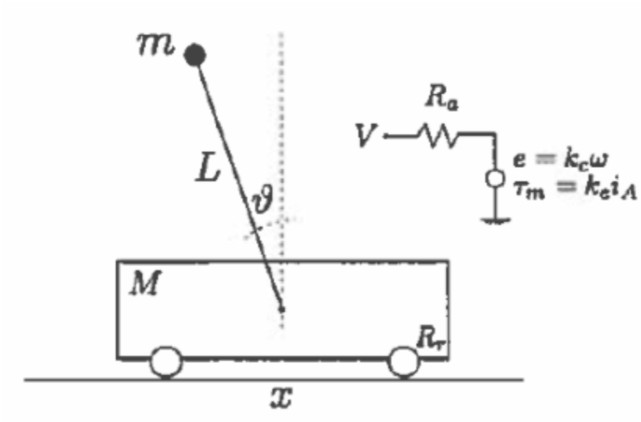

Figure 1: Cart-pendulum system.

Assume that only the angular and linear positions are measured. By discretizing, with sampling period $T = 0.1s$, the model obtained through linearization about the equilibrium corresponding to the vertical position, one obtains

$$\begin{aligned} x^+ &= \underbrace{\begin{bmatrix} 1 & 0.0824 & -0.0087 & -0.0003 \\ 0 & 0.6692 & -0.1652 & -0.0087 \\ 0 & 0.0178 & 1.0582 & 0.1019 \\ 0 & 0.3367 & 1.1652 & 1.0582 \end{bmatrix}}_{A_d} x + \underbrace{\begin{bmatrix} 0.0009 \\ 0.0165 \\ -0.0009 \\ -0.0168 \end{bmatrix}}_{B_d} u \\ y &= \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}}_{C_d} x. \end{aligned} \quad (1)$$

(a) Check that the system is observable and design a Luenberger observer such that the observer eigenvalues are in $e^{-9.7T}$, $e^{-9.8T}$, $e^{-9.9T}$, and e^{-10T}.

Hint: Relevant MATLAB commands: `obsv`, `place`.

Simulate the system for $x(0) = \begin{bmatrix} 0 & 0 & 10^{-4} & 0 \end{bmatrix}^T$ and $u = 0$, and plot the estimation error when $\hat{x}(0) = \begin{bmatrix} 0 & 0 & 0.1 & 0 \end{bmatrix}^T$.

- For simulation, first use MATLAB to simulate the discrete-time system given in (1) as well as the Luenberger observer that you designed. Observe that, despite the fact that the system is unstable, the estimation error $\hat{e}(k) = x(k) - \hat{x}(k)$ converges to zero.
  Hint: Set the simulation time to $t_{sim} = 2s$. Note that the system in (1) is unstable and running the simulation for too long will produce very large output and state values. In such a case, the estimation errors will also increase due to the limited precision of numbers on MATLAB. This can usually be problematic when doing numerical analysis of unstable systems, in general.
- Now test your observer on the continuous nonlinear system dynamics in Simulink. To do that, first run the file *pendulum_sys_init.m* to initialize the simulation parameters. Afterwards, design your Luenberger observer and make sure to have the observer matrix L in the MATLAB workspace. Then, open and simulate the Simulink file *pendulum_openloop_Luenberger.mdl* and look at the plots that are given in the scope blocks. Observe that there is a nonzero steady-state estimation error. What might be the reason for this behavior?

(b) Design a reduced-order observer with eigenvalues in $e^{-9.9T}$ and e^{-10T} and perform the same simulations of the previous point, following the same instructions. In this case, start the estimator $\hat{w}(k)$ dynamics with an initial state of $\begin{bmatrix} 0 & 0.1 \end{bmatrix}^T$.

- To simulate with Simulink, now use the file *pendulum_openloop_reducedorder.mdl* instead. Moreover, before running the Simulink file, make sure that the matrices T, $\bar{A}_{11}$, $\bar{A}_{12}$, $\bar{A}_{21}$, $\bar{A}_{22}$, $\bar{B}_1$, $\bar{B}_2$, and L are in the MATLAB workspace, named as `T`, `Abar11`, `Abar12`, `Abar21`, `Abar22`, `Bbar1`, `Bbar2`, and `L`, respectively.

3. Consider the system

$$\begin{aligned} \bar{x}^+ &= \bar{A}\bar{x} + \bar{B}u \\ y &= \bar{C}\bar{x} \end{aligned}$$

where $\bar{x} \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$ $p < n$, and

$$\bar{x} = \begin{bmatrix} y \\ \bar{x}_2 \end{bmatrix}, \quad \bar{x}_2 \in \mathbb{R}^{n-p}, \quad \bar{A} = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} \bar{B}_1 \\ \bar{B}_2 \end{bmatrix}, \quad \bar{C} = \begin{bmatrix} I & 0 \end{bmatrix}$$

This is the form, used in the lectures, for defining the reduced-order observer (with no delay)

$$\begin{aligned} \hat{x}_2(k+1) &= \bar{A}_{22}\hat{x}_2(k) + \left(\bar{A}_{21}y(k) + \bar{B}_2u(k)\right) \\ &\quad - L\left(y(k+1) - \bar{A}_{11}y(k) - \bar{B}_1u(k) - \bar{A}_{12}\hat{x}_2(k)\right) \\ \hat{x}(k) &= \begin{bmatrix} y(k) \\ \hat{x}_2(k) \end{bmatrix} \end{aligned} \tag{2}$$

Show that, if $(\bar{A}_{22}, \bar{A}_{12})$ is observable and $(\bar{A}, \bar{B})$ is reachable, by using (2) and $u(k) = K\hat{x}(k)$, all eigenvalues of the closed-loop system can be assigned by choosing the matrix gains K and L independently from each other.

Hint: The closed-loop system state is $\begin{bmatrix} \bar{x}_1 & \bar{x}_2 & \hat{x}_2 \end{bmatrix}^T$. Show that, through a suitable change of coordinates, one can use the state $\begin{bmatrix} \bar{x}_1 & \bar{x}_2 & \hat{e}_2 \end{bmatrix}^T$ where $\hat{e}_2 = \bar{x}_2 - \hat{x}_2$. Then, make the error $\hat{e}_2$ appear in the part of the system dynamics governing $\bar{x}_1$ and $\bar{x}_2$.