Multivariable Control (ME-422) - Exercise session 5 SOLUTIONS

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1. Consider the LTI CT system

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

As seen in the lectures, through Backward Euler (BE) discretization with sampling period $T > 0$ one obtains the DT system

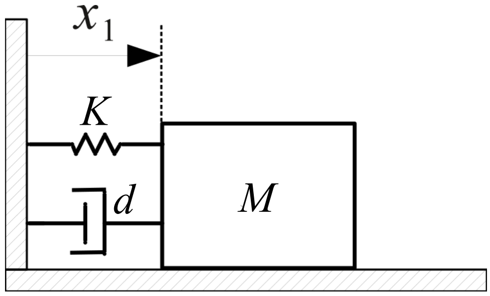
$$x_{k+1} = ATx_{k+1} + x_k + BTu_{k+1} \tag{1}$$
$$y_k = Cx_k + Du_k \tag{2}$$

(a) Show that, defining as a new state $w_k = x_k - TAx_k - TBu_k$, (1)-(2) become

$$w_{k+1} = (I - AT)^{-1}w_k + (I - AT)^{-1}BTu_k$$
$$y_k = C(I - AT)^{-1}w_k + \left[D + C(I - AT)^{-1}BT\right]u_k$$

where we assumed that $I - AT$ is invertible.

(b) Consider the mass-spring-damper system



represented by the CT model

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ -\frac{K}{M} & -\frac{d}{M}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ \frac{1}{M}\end{bmatrix}u$$

where $M = 1$, $K \geq 0$, and $d \geq 0$.

In the Exercise session 3, we have analyzed the DT model

$$\begin{bmatrix}x_1^+\\ x_2^+\end{bmatrix} = \begin{bmatrix}1 & T\\ -\frac{K}{M}T & 1 - \frac{d}{M}T\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ \frac{T}{M}\end{bmatrix}u,$$

produced by Forward Euler (FE) discretization with $T > 0$.

Compute the DT model obtained through BE discretization and analyze the stability properties in the following cases

- $T = 1$, $K = 0$, $d = 2$
- $T = 1$, $K = 0.5$, $d = 0$

Are the results different from those obtained for FE discretization? Which discretized model better captures the physical properties of the system?

Solution:

(a) The definition of w_k can be exploited to write x_k in terms of w_k and u_k such as

$$x_k - TAx_k - TBu_k \triangleq w_k \implies (I - AT)x_k = w_k + TBu_k$$

and one has

$$x_k = (I - AT)^{-1}w_k + (I - AT)^{-1}TBu_k \tag{3}$$

assuming that $I - AT$ is invertible.

Then, arranging equation (1) to have all elements with index $k+1$ in the left-hand side and leave x_k on the right-hand side, we have

$$x_{k+1} - TAx_{k+1} - TBu_{k+1} \triangleq w_{k+1} = x_k$$

Merging the last two equations, one gets the equation for the dynamics of w_k as

$$w_{k+1} = (I - AT)^{-1}w_k + (I - AT)^{-1}BTu_k$$

In the output equation, replacing the expression for x_k in (3), one gets

$$\begin{aligned} y_k &= C\left((I - AT)^{-1}w_k + (I - AT)^{-1}TBu_k\right) + Du_k \\ &= C(I - AT)^{-1}w_k + \left(D + C(I - AT)^{-1}TBu_k\right) \end{aligned}$$

(b) As shown in the previous part of the exercise, discretized dynamics of this system using BE discretization can be expressed as

$$\begin{aligned} w_{k+1} &= (I - AT)^{-1}w_k + (I - AT)^{-1}BTu_k \\ y_k &= C(I - AT)^{-1}w_k + \left[D + C(I - AT)^{-1}BT\right]u_k \end{aligned}$$

where $A = \begin{bmatrix} 0 & 1 \\ -K & -d \end{bmatrix}$ and $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Taking $T = 1$, one can calculate

$$(I - AT)^{-1} = \begin{bmatrix} 1 & -1 \\ K & 1+d \end{bmatrix}^{-1} = \frac{1}{1+d+K}\begin{bmatrix} 1+d & 1 \\ -K & 1 \end{bmatrix}$$

i. With $K = 0$ and $d = 2$, we have $(I - AT)^{-1} = \begin{bmatrix} 1 & \frac{1}{3} \\ 0 & \frac{1}{3} \end{bmatrix}$ and its eigenvalues are

$$\begin{aligned} &\lambda_1 = 1 \qquad |\lambda_1| = 1 \qquad n_1 = 1 = \eta_1 \implies \text{Stable mode} \\ &\lambda_2 = \frac{1}{3} \qquad |\lambda_2| < 1 \implies \text{Stable mode} \end{aligned}$$

Therefore, the system is stable with this choice of parameters.

ii. With $K = 0.5$ and $d = 0$, we have $(I - AT)^{-1} = \begin{bmatrix} \frac{2}{3} & \frac{2}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{bmatrix}$ and its eigenvalues are

$$\lambda_{1,2} = 0.6667 \pm 0.4714i \qquad |\lambda_{1,2}| = 0.8165 < 1 \implies \text{Stable mode}$$

Therefore, the system is stable with this choice of parameters.

We see that BE discretization preserves stability in this case, whereas FE discretization done in Exercise session 3 did not. Therefore, it can be concluded that BE discretization better captures the physical properties of the system compared to FE.

2. Assume that the DT LTI system

$$x^+ = Ax + Bu \qquad x \in \mathbb{R}^n, u \in \mathbb{R}^m$$

is reachable and let $M_t = \begin{bmatrix} B & AB & \dots & A^{t-1}B \end{bmatrix}$ be the t-step rechability matrix.

For a given $\bar{x} \in \mathbb{R}^n$, among all inputs $u(k)$, $k = 0, \dots, t-1$ that steer $x(0) = 0$ to $x(t) = \bar{x}$, the one that minimizes

$$\sum_{k=0}^{t-1} \|u(k)\|^2$$

is given by

$$\begin{bmatrix} u_{ln}(t-1) \\ \vdots \\ u_{ln}(0) \end{bmatrix} = M_t^T \left(M_t M_t^T \right)^{-1} \bar{x} \tag{4}$$

The input u_{ln} is called *least-norm* (or *minimum-energy*) input.

(a) Show that $u_{ln}(k) = B^T (A^T)^{t-1-k} \left(\sum_{s=0}^{t-1} A^s B B^T \left(A^T\right)^s \right)^{-1} \bar{x}$ for $k = 0, \dots, t-1$.
Hint: Use the block structure of M_t for calculating the products in (4).

(b) Show that the minimum energy $\mathcal{E}_{min}(\bar{x}, t) = \sum_{k=0}^{t-1} \|u_{ln}\|^2$ required to reach $x(t) = \bar{x}$ is

$$\mathcal{E}_{min}(\bar{x}, t) = \bar{x}^T G_t^{-1} \bar{x}$$
$$G_t = \sum_{k=0}^{t-1} A^k B B^T \left(A^T\right)^k$$

The matrix G_t is called "t-steps reachability Gramian".

(c) $\mathcal{E}_{min}(\bar{x}, t)$ measures how hard it is to reach $x(t) = \bar{x}$ from $x(0) = 0$. In particular, the ellipsoid

$$\{\bar{x} \in \mathbb{R}^n : \mathcal{E}_{min}(\bar{x}, t) \leq 1\}$$

shows states reachable (in t steps) with one unit of energy. It also shows directions that can be reached with small inputs and directions that can be reached only with large inputs. Plot the ellipsoid in the state space for $t = 3$ and $t = 10$ when

$$A = \begin{bmatrix} 1.75 & 0.8 \\ -0.95 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{5}$$

(d) Using the data (5) plot $\mathcal{E}_{min}(\bar{x}, t)$as a function of t, with $\bar{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

(e) For general pairs (A, B) and target states $\bar{x}$, one has

$$t \geq s \implies \mathcal{E}_{min}(\bar{x}, t) \leq \mathcal{E}_{min}(\bar{x}, s) \tag{6}$$

i.e., it takes less energy to reach $\bar{x}$ over bigger time horizons.
Show that relation (6) holds when $A \in \mathbb{R}$ and $B \in \mathbb{R}$.

Solution:

(a) We can calculate the explicit expression of *least-norm* input sequence by considering the block structure of $M_t = \begin{bmatrix} B & AB & \dots & A^{t-1}B \end{bmatrix}$ as

$$
\begin{aligned}
\begin{bmatrix} u_{ln}(t-1) \\ \vdots \\ u_{ln}(0) \end{bmatrix} &= M_t^T \left(M_t M_t^T\right)^{-1} \bar{x} \\
&= \begin{bmatrix} B^T \\ B^T A^T \\ \vdots \\ B^T \left(A^{t-1}\right)^T \end{bmatrix} \left(\begin{bmatrix} B & AB & \dots & A^{t-1}B \end{bmatrix} \begin{bmatrix} B^T \\ B^T A^T \\ \vdots \\ B^T \left(A^{t-1}\right)^T \end{bmatrix} \right)^{-1} \bar{x} \\
&= \begin{bmatrix} B^T \\ B^T A^T \\ \vdots \\ B^T \left(A^{t-1}\right)^T \end{bmatrix} \left(BB^T + ABB^T A^T + \dots + A^{t-1}BB^T \left(A^{t-1}\right)^T \right)^{-1} \bar{x} \\
&= \begin{bmatrix} B^T \\ B^T A^T \\ \vdots \\ B^T \left(A^{t-1}\right)^T \end{bmatrix} \left(\sum_{s=0}^{t-1} A^s BB^T \left(A^s\right)^T \right)^{-1} \bar{x}
\end{aligned}
$$

Then, we can see that each element $u_{ln}(k)$ of input sequence is written as

$$
u_{ln}(k) = B^T (A^T)^{t-1-k} \left(\sum_{s=0}^{t-1} A^s BB^T \left(A^T\right)^s \right)^{-1} \bar{x}
$$

for $k = 0, \dots, t-1$.

(b) Writing the expression for the minimum energy, we get

$$
\begin{aligned}
\mathcal{E}_{min}(\bar{x}, t) = \sum_{k=0}^{t-1} \|u_{ln}\|^2 = \begin{bmatrix} u_{ln}(t-1) \\ \vdots \\ u_{ln}(0) \end{bmatrix}^T \begin{bmatrix} u_{ln}(t-1) \\ \vdots \\ u_{ln}(0) \end{bmatrix} &= \left(M_t^T \left(M_t M_t^T\right)^{-1} \bar{x} \right)^T M_t^T \left(M_t M_t^T\right)^{-1} \bar{x} \\
= \bar{x}^T \left(M_t M_t^T\right)^{-1} M_t M_t^T \left(M_t M_t^T\right)^{-1} \bar{x} &= \bar{x}^T \left(M_t M_t^T\right)^{-1} \bar{x} \\
= \bar{x}^T \left(\sum_{k=0}^{t-1} A^k BB^T \left(A^T\right)^k \right)^{-1} \bar{x}
\end{aligned}
$$

(c) See the MATLAB file `Ex5.m`.

(d) See the MATLAB file `Ex5.m`.

(e) For $A \in \mathbb{R}$, $B \in \mathbb{R}$, and $t \geq s$, we have that

$$
\sum_{k=0}^{t-1} A^k BB^T \left(A^T\right)^k = \sum_{k=0}^{t-1} A^{2k} B^2 = \sum_{k=0}^{s-1} A^{2k} B^2 + \underbrace{\sum_{k=s}^{t-1} A^{2k} B^2}_{\geq 0} \geq \sum_{k=0}^{s-1} A^{2k} B^2 = \sum_{k=0}^{s-1} A^k BB^T \left(A^T\right)^k
$$

hence

$$
\left(\sum_{k=0}^{t-1} A^k BB^T \left(A^T\right)^k \right)^{-1} \leq \left(\sum_{k=0}^{s-1} A^k BB^T \left(A^T\right)^k \right)^{-1}
$$

Therefore, it satisfies that $\mathcal{E}_{min}(\bar{x}, t) \leq \mathcal{E}_{min}(\bar{x}, s)$.