Multivariable Control (ME-422) - Exercise session 4 SOLUTIONS

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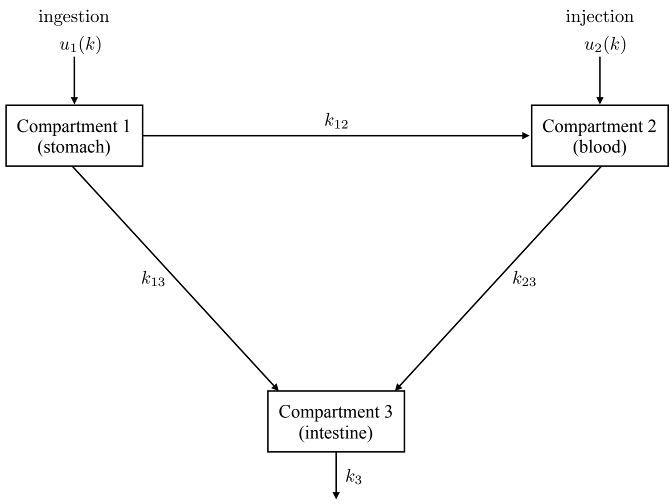
1. **Obersvability analysis**
   Consider the following multi-compartment system, which is similar to the one analyzed in the last exercise session



where the state $x_i(k)$ of each compartment is the mass of the drug (in mg) in compartment i, the mass-transfer rates are $k_{12} = k_{13} = k_3 = 0.5h^{-1}$, $k_{23} > 0$, and inputs are measured in mg/h. Assume the output is the drug concentration in compartment 3 (with volume $V = 2cc$), a discrete-time model of the system is

$$\begin{cases} x_1^+ = -(k_{12} + k_{13})x_1 + x_1 + u_1 \\ x_2^+ = k_{12}x_1 - k_{23}x_2 + x_2 + u_2 \\ x_3^+ = k_{13}x_1 + k_{23}x_2 - k_3x_3 + x_3. \end{cases}$$

$$y = \frac{1}{V}x_3.$$

(a) Set $k_{23} = 0$. Using MATLAB, verify that the system is unobservable. Can you guess an initial state $x(0) \neq 0$ producing a zero output by just looking at the system structure?

Verify your guess by simulating free state of the system.

(b)
- Set $k_{23} = 0.5$ and verify that the system is still unobservable. This shows that unobservability cannot be always deduced just by looking at the system structure.
- Compute the unobservable eigenvalue using the PBH test.
- Build the observability form of the system and verify the result in the previous part.

- Find an unobservable state and simulate the results.

(c) Set $k_{23} = 1$ and verify that the system is observable. Reconstruct the initial state producing the output

$$y(0) = 0 \qquad y(1) = 0.75 \qquad y(2) = 0.625$$

Solution: See the MATLAB file `Ex4.m`.

2. **Observability and reconstructibility**
An LTI system $x^+ = Ax \quad y = Cx \quad x \in \mathbb{R}^n$ is reconstructible if the measurements $y(k)$ $k = 0, 1, \ldots, n-1$ allow one to determine x_{n-1} uniquely.

(a) Show that (A, C) observable $\implies$ (A, C) reconstructible.

(b) Assume that $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ $\quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}$. Is the system observable? Is it reconstructible?

(c) Show that if (A, C) is reconstructible and $\det(A) \neq 0$, then (A, C) is also observable.

Solution:

(a) One has $x(n-1) = A^{n-1}x(0)$. Moreover, if (A, C) is observable, there is only one solution $x(0)$ of the linear system

$$M_o^T x(0) = y$$

where $y = [y^T(0), \ldots, y^T(n-1)]^T$. Therefore, $x(n-1)$ can be uniquely determined from the output.

(b) The system is unobservable because it is in the observability form

$$A = \begin{bmatrix} \hat{A}_a & 0 \\ \hat{A}_{ba} & \hat{A}_b \end{bmatrix} \quad C = \begin{bmatrix} \hat{C}_a & 0 \end{bmatrix}$$

which reveals the existence of an unobservable part.
The state evolution is

$$x(0) = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} \qquad y(0) = Cx(0) = x_1(0)$$
$$x(1) = Ax(0) = \begin{bmatrix} 0 \\ x_1(0) \end{bmatrix} \qquad y(1) = Cx(1) = 0$$

hence $x(1)$ can be always reconstructed from the knowledge of $y(0)$ and $y(1)$. The system is reconstructible.

(c) If (A, C) is reconstructible, then $x(n-1)$ can be uniquely determined from $y = \begin{bmatrix} y^T(0), \ldots, y^T(n-1) \end{bmatrix}^T$. One also has $x(n-1) = A^{n-1}x(0)$. Since A is invertible

$$x(0) = (A^{-1})^{n-1} x(n-1)$$

and hence $x(0)$ can be uniquely determined from y.

3. A SISO LTI system

$$x^+ = Ax \quad y = Cx$$

is in the observable canonical form if

$$A = \begin{bmatrix} -a_1 & 1 & 0 & \ldots & 0 \\ -a_2 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_{n-1} & 0 & 0 & \ldots & 1 \\ -a_n & 0 & 0 & \ldots & 0 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 & \ldots & 0 \end{bmatrix}$$

Show that the system is always observable.

Solution: Note that the observability matrix for the pair (A, C) is $M_o = \begin{bmatrix} C^T & A^TC^T & \dots & (A^T)^{n-1}C^T \end{bmatrix}$. Then, with the specific choice of A and C matrices above, we have

$$C^T = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad A^TC^T = \begin{bmatrix} -a_1 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad (A^T)^2C^T = \begin{bmatrix} a_1^2 - a_2 \\ -a_1 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Consequently, one can see that every vector $(A^T)^{k-1}C^T$ $k = 1, 2, \dots, n$ has 1 at its kth element. Therefore, the main diagonal of the observability matrix M_o consists of elements equal to 1. Following this, M_o is full rank, and therefore the system is observable.