

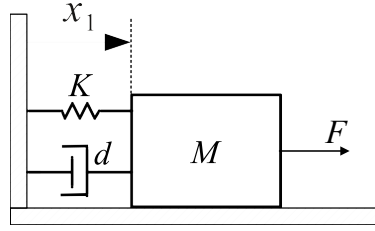
Multivariable Control (ME-422) - Exercise session 3

SOLUTIONS

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1. Stability analysis

Consider the mass-spring-damper system



with $M > 0$, $K \geq 0$, and $d \geq 0$.

As seen in the lecture, a discrete-time model of the system is

$$\begin{bmatrix} x_1^+ \\ x_2^+ \end{bmatrix} = \begin{bmatrix} 1 & T \\ -\frac{K}{M}T & 1 - \frac{d}{M}T \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{T}{M} \end{bmatrix} u,$$

where $T > 0$ is the sampling time, $u = F$, and x_1 and x_2 are, respectively, the position and velocity of the mass. Set $T = 1$, $M = 1$, and, for each one of the following cases, analyze the stability of the system and compute the modes.

- $K = 0$, $d = 0$
- $K = 0$, $d \in (0, 1)$
- $K \in (0, 1)$, $d = 0$

In each case, check if the results match the physical intuition about the motion of the system.

Solution:

- $K = 0$, $d = 0$:

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \rightarrow \text{eigenvalue 1 with algebraic multiplicity 2}$$

The system can be either stable or unstable. For deciding, one has to compute the geometric multiplicity of the eigenvalue 1.

$$V_1 = \ker(A - I) = \ker \left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right) = \left\{ x \in \mathbb{R}^2 : \begin{matrix} x_2 = 0 \\ 0 = 0 \end{matrix} \right\} = \left\{ \begin{bmatrix} \alpha \\ 0 \end{bmatrix}, \alpha \in \mathbb{R} \right\} \rightarrow \dim(V_1) = 1$$

Since $\dim(V_1) < 2$, the system is unstable.

For computing the modes, we use the table seen in the lecture. The integer η verifies $1 \leq \eta \leq 2$ and $\eta = 1 \iff n = \nu$. Hence, $\eta = 2$. From the table, we have two modes indexed by $p = 0, 1$

$$m_1(k) = 1^k = 1 \quad (\text{corresponding to } p = 0)$$

$$m_2(k) = \begin{cases} 0, & \text{for } k = 0 \\ k1^{(k-1)}, & \text{for } k > 0 \end{cases} \quad (\text{corresponding to } p = 1)$$

The modes are in agreement with the physical intuition: in absence of damping and elastic forces, if the initial velocity is positive and $F = 0$, the position of the mass diverges. This behavior of the free state is captured by $m_2(k)$.

- $K = 0, d \in (0, 1)$:

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1-d \end{bmatrix} \rightarrow \lambda_1 = 1, \quad \lambda_2 = 1-d$$

The system is stable. The eigenvalues are real and distinct. From the table, the modes are

$$m_1(k) = 1^k = 1, \quad m_2(k) = (1-d)^k.$$

Physical intuition: under the effect of damping only, the free state starting from $x_1 = 0, x_2 \neq 0$ is expected to converge to a position $\bar{x}_1 \neq 0$. However, this should happen for any $d > 0$ and not only for $d \in (0, 1)$. This discrepancy between the mathematical analysis and physical intuition is due to the fact that discrete-time model is just an approximation of the continuous-time one. This phenomenon will be analyzed later in the lectures.

- $K \in (0, 1), d = 0$:

$$A = \begin{bmatrix} 1 & 1 \\ -K & 1 \end{bmatrix}$$

Compute the eigenvalues as

$$\det(\lambda I - A) = \det \left(\begin{bmatrix} \lambda - 1 & -1 \\ K & \lambda - 1 \end{bmatrix} \right) = (\lambda - 1)^2 + K = \lambda^2 - 2\lambda + 1 + K$$

$$\lambda_{1,2} = 1 \pm \sqrt{1 - 1 - K} = 1 \pm j\sqrt{K}.$$

Since $|\lambda_1| = |\lambda_2| > 1$, the system is unstable.

This does not match the physical intuition on the continuous-time system: the free state starting from $x_1 \neq 0, x_2 = 0$ is expected to be bounded, as the mass will oscillate around $\bar{x}_1 = 0$ under the effect of the elastic force. The mismatch is due to the fact that the discretized system is just an approximation of the continuous-time model.

2. Stability analysis

Analyze by hand the stability of the system $x^+ = Ax$ where

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Hint: The eigenvalues of a block-diagonal matrix are the union of the eigenvalues of the blocks on the main diagonal.

Solution: The eigenvalues of A are $\lambda_1 = 0$, $\lambda_2 = 1$, $\lambda_3 = 0$, and $\lambda_4 = 1$. Then, the system cannot be asymptotically stable. It can be stable or unstable. To decide, let us compute the eigenspace

$$V_1 = \ker(A - 1I) = \ker \left(\underbrace{\begin{bmatrix} -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\tilde{A}} \right) = \left\{ x \in \mathbb{R}^4 : \tilde{A}x = 0 \right\}$$

$$= \left\{ x \in \mathbb{R}^4 : \begin{array}{l} -x_1 = 0 \\ -x_1 = 0 \\ -x_3 + 2x_4 = 0 \end{array} \right\} = \left\{ \begin{bmatrix} 0 \\ \alpha \\ 2\beta \\ \beta \end{bmatrix}, \alpha, \beta \in \mathbb{R} \right\}.$$

Since the geometric multiplicity of the eigenvalue 1 is $\dim(V_1) = 2$, and it coincides with the algebraic multiplicity, the system is simply stable.

3. Invariance of reachability

Consider the LTI system

$$x^+ = Ax + Bu \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m.$$

Show that the rank of the reachability matrix is invariant with respect to the change of coordinate $\hat{x} = Tx$, $\det(T) \neq 0$.

This means that the reachability properties of the equivalent LTI systems are the same.

Solution: As seen in the lectures, the system dynamics in the $\hat{x}$ -coordinates is

$$\hat{x}^+ = \hat{A}\hat{x} + \hat{B}u \quad \hat{A} = TAT^{-1}, \quad \hat{B} = TB. \quad (1)$$

The reachability matrix of the system (1) is

$$\hat{M}_r = [\hat{B} \quad \hat{A}\hat{B} \quad \dots \quad \hat{A}^{n-1}\hat{B}].$$

Since $\hat{A}^k \hat{B} = \underbrace{TAT^{-1} \dots TAT^{-1}}_{k \text{ times}} TB = TA^k B$, one has

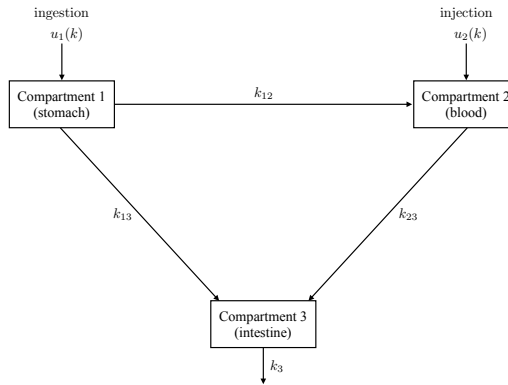
$$\hat{M}_r = [TB \quad TAB \quad \dots \quad TA^{n-1}B] = T \underbrace{[B \quad AB \quad \dots \quad A^{n-1}B]}_{M_r}$$

where M_r is the reachability matrix of (A, B) . Since T is nonsingular, one has $\text{rank}(\hat{M}_r) = \text{rank}(M_r)$.

4. Consider the multi-compartment system describing the discrete-time dynamics of a drug on various parts of the body, given in the figure, where the state $x_i(k)$ of each compartment is the mass of the drug measured in mg in compartment i , and arrows represent mass transfer. The mass transfer rates are $k_{12} = k_{13} = k_{23} = k_3 = 0.5h^{-1}$. Inputs are mass flows measured in mg/h .

A discrete-time model describing the mass transfer between compartments is

$$\begin{aligned} x_1^+ &= -(k_{12} + k_{13})x_1 + x_1 + u_1 \\ x_2^+ &= k_{12}x_1 - k_{23}x_2 + x_2 + u_2 \\ x_3^+ &= k_{13}x_1 + k_{23}x_2 - k_3x_3 + x_3. \end{aligned}$$



- (a)
- Remove the input u_1 from the model. Using MATLAB, check that the system is unreachable.
 - Can you compute an unreachable state just by looking at the system structure? (**Hint:** Think about an experiment starting without drug in the body.)

Build the reachability form of the system and verify your result.

- (b) Add back the input u_1 and check if the system is reachable, using MATLAB. If yes, compute a control law $u(k)$, $k = 0, 1, 2$ that, starting without drug in the body, produces $x_1(3) = 0.5$, $x_2(3) = 1$, and $x_3(3) = 1$.

Solution: See the MATLAB file `Ex3.m`.