Multivariable Control (ME-422) - Exercise session 2B SOLUTIONS

Prof. G. Ferrari Trecate

Consider the Duffing oscillator shown in Figure 1. This system consists of a mass-less flexible arm of length l and a metal ball of mass m. The ball is attached to the upper extreme of the flexible arm and is free to oscillate, whereas the lower extreme of the arm is anchored to the ground. The angular position of the ball can be controlled by generating an electromagnetic torque using two magnets, as shown below.

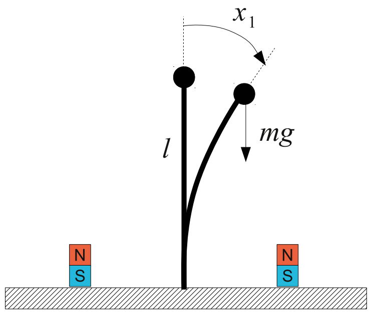


Figure 1: Duffing oscillator

The dynamics of the system can be modeled according to the second-order differential equation:

$$ml^2\ddot{x}_1 = mgl\sin(x_1) - \alpha x_1 - k\dot{x}_1 + \tau\,,$$

where αx_1 represents the restoring torque ($\alpha > 0$) and $k\dot{x}_1$ the damping torque ($k > 0$). Defining $x_2 = \dot{x}_1$ and $u = \frac{\tau}{ml^2}$, we obtain the following state-space model for the Duffing oscillator:

$$\begin{aligned}\dot{x}_1 &= x_2,\\ \dot{x}_2 &= \frac{g}{l}\sin(x_1) - \frac{\alpha}{ml^2}x_1 - \frac{k}{ml^2}x_2 + u.\end{aligned} \tag{1}$$

The values of the parameters are $l = 1\,\mathrm{m}$, $\alpha = 16\,\mathrm{N\,m}$, $m = 2\,\mathrm{kg}$, $k = 4\,\mathrm{N\,m\,s}$ and $g = 9.81\,\frac{\mathrm{m}}{\mathrm{s}^2}$. We consider full state observation.

1. Compute all the equilibrium states $(\bar{x}_1, \bar{x}_2)$ of the system when $\bar{u} = 0$.
   Hint: use the `vpasolve` *function to numerically solve a nonlinear equation. You may need to indicate an initial value to the solver.*

2. Linearize the system around the equilibrium states obtained in the previous point. Are these linearized systems stable? Can you conclude anything about the stability of the equilibria of the original non-linear system?

3. Construct the non-linear model (1) in Simulink. Using the time-based linearization block, compute the matrices that describe the dynamics of the linearized systems around the different equilibria. Compare the results obtained numerically with the analytical solutions obtained in point 2.
   To linearize a non-linear system using Simulink, follow these steps:

   (a) *Replace the input and output signals of the non-linear system with the* `In1` *and* `Out1` *blocks available in the Simulink library under the sources and sinks tabs.*

(b) *Add a* ***Timed-Based Linearization*** *block available in the Simulink library under the model-wide utilities tab.*

(c) *Save the Simulink model as* `filename.slx` *and simulate the evolution of the system. Then, extract the matrices that characterize the linearized system dynamics from the Matlab structure named* `filename_Timed_Based_Linearization` *that will be created automatically.*

Do not forget to properly set the linearization time of the ***Timed-Based Linearization*** *block and the initial condition of the system.*

4. Simulate and compare the evolution of the non-linear system with that of the three linearized systems constructed in the previous point. First, assume that $(x_1(0), x_2(0)) = (0, 0)$. Then, repeat the simulation with initial conditions $(0.5, 0)$ and $(-0.5, 0)$. Plot the state trajectories in the x_1-x_2 plane.
Hint: set the input to zero and define the different initial conditions as workspace variables.

Solution:

1. To calculate the equilibria of the system, we impose $\dot{x}_1 = \dot{x}_2 = 0$ in (1). The first equation implies that $\bar{x}_2 = 0$. Recalling that $\bar{u} = 0$, the second equation gives $9.81 \sin(\bar{x}_1) = 8\bar{x}_1$. We solve the nonlinear equation numerically using Matlab (see file `Ex2B.mlx`). Then, the system has three equilibria at:

   a. $(\bar{x}_1^1, \bar{x}_2^1) = (0, 0)$, b. $(\bar{x}_1^2, \bar{x}_2^2) \approx (1.08, 0)$, c. $(\bar{x}_1^3, \bar{x}_2^3) \approx (-1.08, 0)$.

2. Let $u = \bar{u} + \delta u$, $x_1 = \bar{x}_1^i + \delta x_1^i$ and $x_2 = \bar{x}_2^i + \delta x_2^i$ for $i = 1, 2, 3$. The linearized system dynamics around the equilibria computed in the previous point are given by

$$\begin{aligned} \delta\dot{x}_1^i &= \delta x_2^i, \\ \delta\dot{x}_2^i &= 9.81 \cos(\bar{x}_1^i)\delta x_1^i - 8\delta x_1^i - 2\delta x_2^i + \delta u. \end{aligned}$$

   Rewriting these equations in matrix form, we obtain:

$$\begin{bmatrix} \delta\dot{x}_1^i \\ \delta\dot{x}_2^i \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 9.81\cos(\bar{x}_1^i) - 8 & -2 \end{bmatrix} \begin{bmatrix} \delta x_1^i \\ \delta x_2^i \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \delta u\,,$$

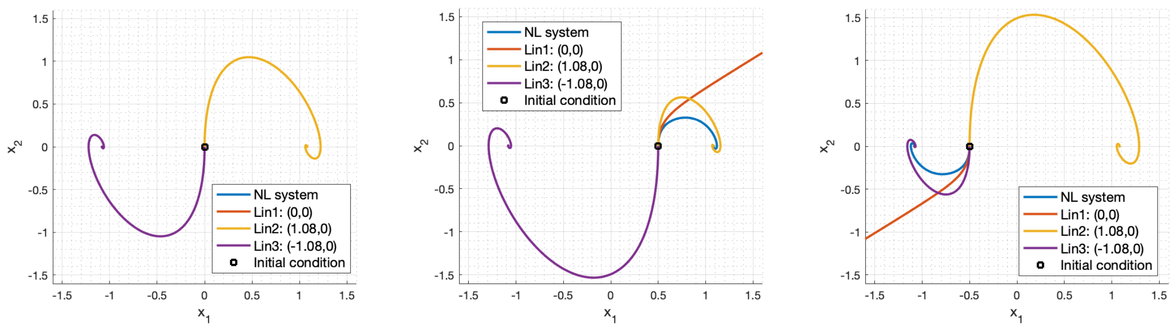
   for $i = 1, 2, 3$. To analyze stability properties, we compute the eigenvalues of the linearized system dynamics around the different equilibrium points. These are approximately given by:

   a. $(\lambda_1^1, \lambda_2^1) = (0.68, -2.68)$, b. $(\lambda_1^2, \lambda_2^2) = (\lambda_1^3, \lambda_2^3) = (-1 + 1.54j, -1 - 1.54j)$.

   Hence, we conclude that the linearized system around $(\bar{x}_1^1, \bar{x}_2^1) = (0, 0)$ is unstable, whereas the systems obtained by linearizing around $(\bar{x}_1^2, \bar{x}_2^2) \approx (1.08, 0)$ and $(\bar{x}_1^3, \bar{x}_2^3) \approx (-1.08, 0)$ are stable. As no eigenvalue has zero real part, we conclude that the equilibrium $(\bar{x}_1^1, \bar{x}_2^1) = (0, 0)$ of the original non-linear system is an unstable equilibrium, whereas $(\bar{x}_1^2, \bar{x}_2^2) \approx (1.08, 0)$ and $(\bar{x}_1^3, \bar{x}_2^3) \approx (-1.08, 0)$ are stable equilibria of the original non-linear system.

3. See the file `Ex2B.mlx`.

4. The linearized dynamics around $(\bar{x}_1^2, \bar{x}_2^2) \approx (1.08, 0)$ and $(\bar{x}_1^3, \bar{x}_2^3) \approx (-1.08, 0)$ are globally asymptotically stable. Hence, regardless of where any of these two linearized system is initialized, it converges back to the equilibrium $(\delta x_1, \delta x_2) = (0, 0)$. Conversely, the linearized dynamics around $(\bar{x}_1^1, \bar{x}_2^1) \approx (0, 0)$ are unstable, and the state trajectories diverge unless the system is perfectly initialized at the equilibrium. Finally, the original non-linear system also remains indefinitely at the unstable equilibrium if it is perfectly initialized there. For general initial conditions, we observe that the dynamics of the non-linear system converge to one of the two stable equilibria.



(a) $(x_1(0), x_2(0)) = (0, 0)$ (b) $(x_1(0), x_2(0)) = (0.5, 0)$ (c) $(x_1(0), x_2(0)) = (-0.5, 0)$

Figure 2: Comparison between the state-space trajectories of the original non-linear system and the three systems obtained by linearization around different equilibrium points, as shown in the legend.