Exercise session 2B

Linearization around an equilibrium and stability

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[1]Dependable Control and Decision Group
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Linearization around an equilibrium

Let $(\bar{x}, \bar{u})$ be an equilibrium for the NL invariant system

$$\dot{x} = f(x, u)$$
$$y = g(x, u)$$

Deviations: $\delta x(t) = x(t) - \bar{x}, \ \delta u(t) = u(t) - \bar{u}, \ \delta y(t) = y(t) - \bar{y}$

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First order Taylor expansion about the equilibrium:

$$f(x, u) \simeq f(\bar{x}, \bar{u}) + D_x f(x, u)\Big|_{\substack{x=\bar{x} \\ u=\bar{u}}}(x - \bar{x}) + D_u f(x, u)\Big|_{\substack{x=\bar{x} \\ u=\bar{u}}}(u - \bar{u})$$

$$g(x, u) \simeq g(\bar{x}, \bar{u}) + D_x g(x, u)\Big|_{\substack{x=\bar{x} \\ u=\bar{u}}}(x - \bar{x}) + D_u g(x, u)\Big|_{\substack{x=\bar{x} \\ u=\bar{u}}}(u - \bar{u})$$

$$D_x f(x, u) = \begin{bmatrix} \frac{\partial f_1(x,u)}{\partial x_1} & \cdots & \frac{\partial f_1(x,u)}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n(x,u)}{\partial x_1} & \cdots & \frac{\partial f_n(x,u)}{\partial x_n} \end{bmatrix}$$ Jacobian with respect to the variables x

Linearization around an equilibrium

One gets:

$$\dot{\delta x} = \dot{x} - \dot{\bar{x}} = f(x,u) \simeq \underbrace{f(\bar{x},\bar{u})}_{=0} + D_x f(x,u)\Big|_{\substack{x=\bar{x}\\u=\bar{u}}} \delta x + D_u f(x,u)\Big|_{\substack{x=\bar{x}\\u=\bar{u}}} \delta u$$

$$\delta y = -\bar{y} + y \simeq \underbrace{-g(\bar{x},\bar{u}) + g(\bar{x},\bar{u})}_{=0} + D_x g(x,u)\Big|_{\substack{x=\bar{x}\\u=\bar{u}}} \delta x + D_u g(x,u)\Big|_{\substack{x=\bar{x}\\u=\bar{u}}} \delta u$$

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Linearized system

Defining

$$A = D_x f(x,u)\Big|_{\substack{x=\bar{x}\\u=\bar{u}}},\ B = D_u f(x,u)\Big|_{\substack{x=\bar{x}\\u=\bar{u}}},\ C = D_x g(x,u)\Big|_{\substack{x=\bar{x}\\u=\bar{u}}},\ D = D_u g(x,u)\Big|_{\substack{x=\bar{x}\\u=\bar{u}}}$$

the linearized system around the equilibrium $(\bar{x},\bar{u})$ is

$$\dot{\delta x} = A\delta x + B\delta u$$
$$\delta y = C\delta x + D\delta u$$

Linearization around an equilibrium

We hope state trajectories of the linearized system are good approximations of $x(t) - \bar{x}$ ... but this does not always happen

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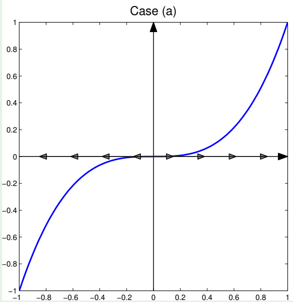
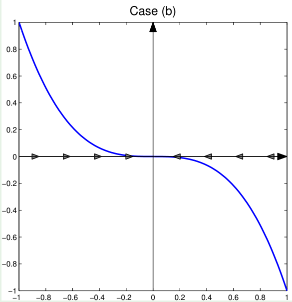
Example: (a): $\dot{x} = x^3$, (b): $\dot{x} = -x^3$

Linearized systems around $\bar{x} = 0$ are the same: $\dot{\delta x} = 0 \Rightarrow \delta x(t) = x_0$ but NL systems have different behaviors





Stability of an equilibrium state of an NL system

Stability of a linear system can be analyzed by looking at the eigenvalues of the transition matrix...

...but how can we analyze the stability
of an equilibrium state of an NL system?[1]

- Stability analysis is difficult in general
- Sufficient conditions for stability of an equilibrium state follow from the dynamics of the linearized system around it

  $\hookrightarrow$ differently from linear systems, stability is a **property of the equilibrium state only**

  $\hookrightarrow$ different equilibrium states of a nonlinear system may have different stability property

[1] Check the Appendix for a mathematical definition of stability of an equilibrium state of a non-linear system.

Stability test for the equilibrium states of an NL system

NL system

$$\mathrm{NL}: \dot{x} = f(x)$$

$\bar{x}$: equilibrium state

Linearized system around $\bar{x}$

$$\mathrm{LIN}: \dot{\delta x} = A(\bar{x})\delta x$$

$$A(\bar{x}) = D_x f(x)\Big|_{x=\bar{x}}$$

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Theorem

The equilibrium state $\bar{x}$ of NL

- is AS if all eigenvalues of LIN have real part < 0
- is unstable if at least an eigenvalue of LIN has real part > 0

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No conclusion if all eigenvalues of LIN have real part ≤ 0 and at least an eigenvalue has zero real part

Appendix

Stability of an equilibrium state

Let $\bar{x}$ be an equilibrium state for the NL invariant system $\dot{x} = f(x)$

Ball centered in $\bar{z} \in \mathbb{R}^n$ of radius $\delta > 0$

$B_\delta(\bar{z}) = \{z \in \mathbb{R}^n : \|z - \bar{z}\| < \delta\}$

Definition (Lyapunov stability)

The equilibrium state $\bar{x}$ is

- stable if

$$\forall \epsilon > 0 \ \exists \delta > 0, \ x(0) \in B_\delta(\bar{x}) \Rightarrow x(t) \in B_\epsilon(\bar{x}), \forall t \geq 0$$

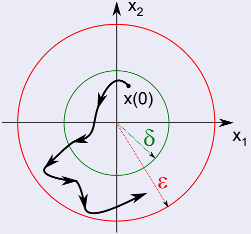
- Asymptotically Stable (AS) if it is stable and $\exists \gamma > 0$ such that

$$x(0) \in B_\gamma(\bar{x}) \Rightarrow \lim_{t \to +\infty} \|x(t) - \bar{x}\| = 0$$

- unstable if it is not stable

Remarks

$\bar{x} = 0$ stable



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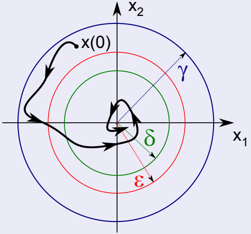
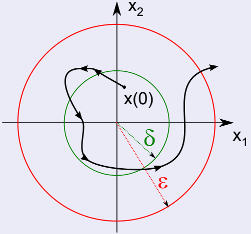


$\bar{x} = 0$ unstable



Regions of attraction of $\bar{x}$ AS

- $X \subseteq \mathbb{R}^n$ is a region of attraction of $\bar{x}$ if

$$x(0) \in X \Rightarrow \lim_{t \to +\infty} \|x(t) - \bar{x}\| = 0$$

Example: $B_\gamma(\bar{x})$ is a region of attraction

- THE region of attraction of $\bar{x}$ is the union of all regions of attraction of $\bar{x}$ (i.e. it is maximal)

Review: stability tests for LTI systems

LTI system

$$\dot{x} = Ax, \quad x(t) \in \mathbb{R}^n$$

System eigenvalues = eigenvalues of the matrix A

Theorem

The equilibrium state $\bar{x} = 0$ of a linear system is

- AS $\Leftrightarrow$ all system eigenvalues have real part < 0
- unstable if at least a system eigenvalue has real part > 0
- stable if all system eigenvalues have real part ≤ 0, at least one has zero real part and all eigenvalues with zero real part are simple

When all eigenvalues have real part ≤ 0 and there are multiple eigenvalues with zero real part, the equilibrium state can be either stable or unstable.