Multivariable Control (ME-422) - Exercise session 2B

Prof. G. Ferrari Trecate

Consider the Duffing oscillator shown in Figure 1. This system consists of a mass-less flexible arm of length l and a metal ball of mass m. The ball is attached to the upper extreme of the flexible arm and is free to oscillate, whereas the lower extreme of the arm is anchored to the ground. The angular position of the ball can be controlled by generating an electromagnetic torque using two magnets, as shown below.

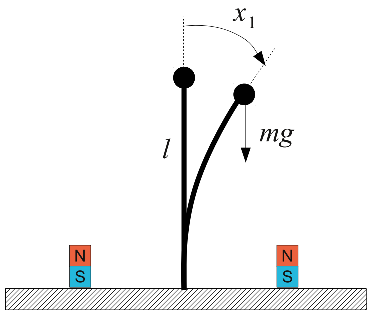

Figure 1: Duffing oscillator

The dynamics of the system can be modeled according to the second-order differential equation:

$$ml^2\ddot{x}_1 = mgl\sin(x_1) - \alpha x_1 - k\dot{x}_1 + \tau\,,$$

where αx_1 represents the restoring torque ($\alpha > 0$) and $k\dot{x}_1$ the damping torque ($k > 0$). Defining $x_2 = \dot{x}_1$ and $u = \dfrac{\tau}{ml^2}$, we obtain the following state-space model for the Duffing oscillator:

$$\begin{aligned}\dot{x}_1 &= x_2,\\ \dot{x}_2 &= \frac{g}{l}\sin(x_1) - \frac{\alpha}{ml^2}x_1 - \frac{k}{ml^2}x_2 + u.\end{aligned} \tag{1}$$

The values of the parameters are $l = 1\,\mathrm{m}$, $\alpha = 16\,\mathrm{N\,m}$, $m = 2\,\mathrm{kg}$, $k = 4\,\mathrm{N\,m\,s}$ and $g = 9.81\,\frac{\mathrm{m}}{\mathrm{s}^2}$. We consider full state observation.

1. Compute all the equilibrium states $(\bar{x}_1, \bar{x}_2)$ of the system when $\bar{u} = 0$.
   Hint: use the `vpasolve` *function to numerically solve a nonlinear equation. You may need to indicate an initial value to the solver.*

2. Linearize the system around the equilibrium states obtained in the previous point. Are these linearized systems stable? Can you conclude anything about the stability of the equilibria of the original non-linear system?

3. Construct the non-linear model (1) in Simulink. Using the time-based linearization block, compute the matrices that describe the dynamics of the linearized systems around the different equilibria. Compare the results obtained numerically with the analytical solutions obtained in point 2.
   To linearize a non-linear system using Simulink, follow these steps:

   (a) *Replace the input and output signals of the non-linear system with the* `In1` *and* `Out1` *blocks available in the Simulink library under the sources and sinks tabs.*

(b) *Add a* ***`Timed-Based Linearization`*** *block available in the Simulink library under the model-wide utilities tab.*

(c) *Save the Simulink model as* `filename.slx` *and simulate the evolution of the system. Then, extract the matrices that characterize the linearized system dynamics from the Matlab structure named* `filename_Timed_Based_Linearization` *that will be created automatically.*

Do not forget to properly set the linearization time of the ***`Timed-Based Linearization`*** *block and the initial condition of the system.*

4. Simulate and compare the evolution of the non-linear system with that of the three linearized systems constructed in the previous point. First, assume that $(x_1(0), x_2(0)) = (0, 0)$. Then, repeat the simulation with initial conditions $(0.5, 0)$ and $(-0.5, 0)$. Plot the state trajectories in the x_1-x_2 plane.
Hint: set the input to zero and define the different initial conditions as workspace variables.