Multivariable Control (ME-422) - Exercise session 2 SOLUTIONS

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1. **Exercise on modal analysis**
   Consider the system $x^+ = Ax$ where

$$A = \begin{bmatrix} -0.5 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & -0.5 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Tell if

(a) all modes go to zero as $k \to \infty$

(b) all modes are bounded

(c) there is at least an unbounded mode.

Hint: Note that A is block diagonal and its eigenvalues are those of each individual block.

Solution: By inspection (block diagonal matrix)

$$\lambda_1 = -0.5 \quad n_1 = 2 \quad \nu_1 = ?$$
$$\lambda_2 = 1 \quad n_2 = 2 \quad \nu_2 = ?.$$

- $|\lambda_1| < 1 \quad \xrightarrow{\text{Lemma}} \quad$ all modes associated to λ_1 are bounded and go to 0, independently of ν_1.
  - **At home:** Show that $\nu_1 = 2$ and hence there is a unique mode associated to λ_1.
- $|\lambda_2| = 1 \quad \xrightarrow{\text{Lemma}} \quad$ bounded or unbounded modes: it depends on ν_2.

$$\nu_2 = dim\,(V_1)$$

$$V_1 = \{v : (A - I)\,v = 0\} = \left\{ \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} : \begin{aligned} -1.5v_1 &= 0 \\ -v_1 &= 0 \\ -1.5v_3 + 2v_4 &= 0 \end{aligned} \right\}$$
$$= \left\{ v : \begin{bmatrix} 0 \\ \alpha \\ \beta \\ \frac{2}{1.5}\beta \end{bmatrix}, \alpha, \beta \in \mathbb{R} \right\} \longrightarrow \nu_2 = 2.$$

Since $n_2 = \nu_2$, the table tells us there is only one mode and the Lemma tells us the mode is bounded.

Conclusion: the answer is (b).

2. **Complex conjugate eigenvalues generate real modes**
At first sight, it might seem strange that complex conjugate eigenvalues give rise to modes taking values in $\mathbb{R}$ instead of $\mathbb{C}$. This exercise provides some intuition on why this happens.
Consider the system

$$\begin{cases} x^+ = Ax \\ y = Cx \end{cases}$$
$$x(0) = x_0.$$

(a) Let $A \in \mathbb{R}^{n\times n}$ and $T \in \mathbb{C}^{n\times n}$ be nonsingular matrices.
Define $\hat{x}(k) = Tx(k) \in \mathbb{C}^n$. Show that $\hat{x}(k)$ follows the dynamics $\hat{x}^+ = \hat{A}\hat{x}$ with $\hat{A} = TAT^{-1}$.

Solution: The relation has been shown in the lectures.

(b) Consider the case

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 1 \end{bmatrix},$$

where the eigenvalues of A are $1 \pm j = \sqrt{2}e^{\pm j\frac{\pi}{4}}$ and

$$\hat{A} = TAT^{-1} = \begin{bmatrix} 1+j & 0 \\ 0 & 1-j \end{bmatrix} \text{ for } T^{-1} = \begin{bmatrix} 1 & 1 \\ j & -j \end{bmatrix}.$$

Compute explicitly $x(k)$ and $y(k)$ as a function of x_0.

Solution:

$$\begin{aligned} x(k) &= 0.5 \begin{bmatrix} 1 & 1 \\ j & -j \end{bmatrix} \begin{bmatrix} \left(\sqrt{2}e^{j\frac{\pi}{4}}\right)^k & 0 \\ 0 & \left(\sqrt{2}e^{-j\frac{\pi}{4}}\right)^k \end{bmatrix} \begin{bmatrix} 1 & -j \\ 1 & j \end{bmatrix} x_0 = \\ &= 0.5\sqrt{2^k} \begin{bmatrix} e^{jk\frac{\pi}{4}} + e^{-jk\frac{\pi}{4}} & -je^{jk\frac{\pi}{4}} + je^{-jk\frac{\pi}{4}} \\ je^{jk\frac{\pi}{4}} - je^{-jk\frac{\pi}{4}} & e^{jk\frac{\pi}{4}} + e^{-jk\frac{\pi}{4}} \end{bmatrix} x_0 = \\ &= \sqrt{2^k} \begin{bmatrix} \cos(k\pi/4) & \sin(k\pi/4) \\ -\sin(k\pi/4) & \cos(k\pi/4) \end{bmatrix} x_0. \end{aligned}$$

From the output equation, we obtain

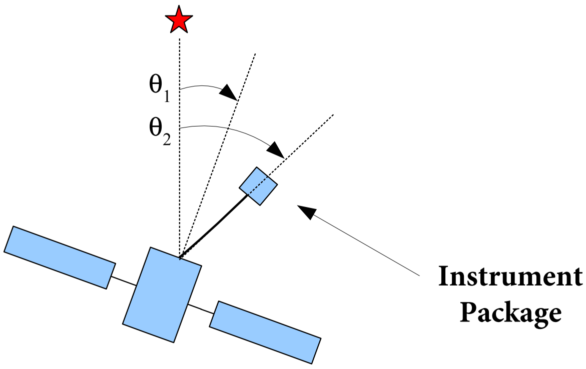
$$\begin{aligned} y(k) &= \sqrt{2^k} \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} \cos(k\pi/4) & \sin(k\pi/4) \\ -\sin(k\pi/4) & \cos(k\pi/4) \end{bmatrix} x_0 = \\ &= \sqrt{2^{k+1}} \begin{bmatrix} \cos((k+1)\pi/4) & \cos((k-1)\pi/4) \end{bmatrix} x_0. \end{aligned}$$

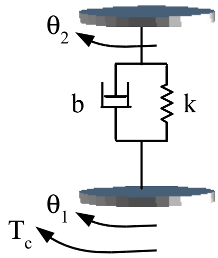
3. **Satellite simulation**
Consider the satellite in the figure. The satellite mission requires accurate pointing of a scientific sensor package with respect to a fixed star.

- θ_1 is the angle deviation of the satellite body
- θ_2 is the angle deviation of the instrument package from the reference star. It can be measured through devices called "star trackers."

By using cold-gas jets, one can produce a torque T_c acting on the satellite body.

The figure also shows an equivalent mechanical model of the satellite. This system is composed by two rotating masses connected by a spring with torque constant κ and viscous-damping constant b.





A discrete-time model of the system with state $x = \begin{bmatrix} \theta_2 & \dot{\theta}_2 & \theta_1 & \dot{\theta}_1 \end{bmatrix}$, output $y = \theta_2$, and input $u = T_c$ can be written as

$$\begin{cases} x^+ = \begin{bmatrix} 1 & T & 0 & 0 \\ -T\frac{\kappa}{J_2} & 1 - T\frac{b}{J_2} & T\frac{\kappa}{J_2} & T\frac{b}{J_2} \\ 0 & 0 & 1 & T \\ T\frac{\kappa}{J_1} & T\frac{b}{J_1} & -T\frac{\kappa}{J_1} & 1 - T\frac{b}{J_1} \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 0 \\ T\frac{1}{J_1} \end{bmatrix} u \\ y = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} x, \end{cases}$$

where $T > 0$ is the sampling time and J_1 and J_2 are the inertias of the masses. Set $\kappa = 0.091$, $b = 0.0036$, $J_1 = 1$, $J_2 = 0.1$, and $T = 0.01$.

(a) Check if the system has unbounded modes. Use MATLAB for computations.

(b) Draw the free output between 0 and $200s$., starting from the initial condition $x = [0 \quad 0 \quad 0.1 \quad 0]^T$. Give a physical interpretation to the result.

(c) Draw the impulse response (that is, the output generated by the input $u(0) = 1,\ u(k) = 0,\ k > 0$) and give an interpretation of the results in terms of the satellite motion.

(d) Now consider the output as $y = x_2$. Draw the impulse response and, through the physical interpretation of the results, explain why its asymptotic value is consistent with the graph obtained in the previous point.

Useful MATLAB commands: `eig`, `null`, `ss(A,B,C,D,T)`, `lsim`, `impulse`.

Solution: See the MATLAB file `Ex2.m`.