Multivariable Control (ME-422) - Exercise session 14

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1. Consider the nonlinear system

$$\begin{aligned} x_{k+1} &= \alpha x_k + \cos(x_k) + w_k & \qquad w_k &\sim WGN(0,1) \\ y_k &= x_k^2 + v_k & \qquad v_k &\sim WGN(0,0.5) \end{aligned}$$

   Determine the Extended Kalman Filter (EKF).

2. Consider the system

$$x_{t+1} = Ax_t + Bu_t \,, \tag{1}$$

   with

$$A = \begin{bmatrix} 0.5 & -2 \\ 6 & -6 \end{bmatrix} , \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} ,$$

   where $x_0 \sim \mathcal{N}(0, \Sigma_0)$ is distributed according to a Gaussian distribution with covariance matrix $\Sigma_0 = I$.

   Since the system is open-loop unstable, your colleague selects a stabilizing control law

$$u_t = -K_0 x_t \,,$$

   that sets the eigenvalues of $(A - BK_0)$ equal to $(0, 0.5)$. Letting

$$J(K) = \mathbb{E}_{x_0}\left[\sum_{t=0}^{\infty} x_t^\top x_t + u_t^\top u_t\right] , \tag{2}$$

   denote the average IH cost achieved by a control law $u_t = -Kx_t$, determine the *Performance Improvement* (PI)

$$\mathrm{PI} = 100\left(1 - \frac{J(K^\star)}{J(K_0)}\right)\% \,, \tag{3}$$

   that can be achieved by replacing K_0 with the optimal S-LQR controller $K^\star$.

   Hint: You can use the MATLAB functions `dlyap` and `idare` appropriately.

3. In this exercise, you will implement the Gradient Descent (GD) algorithm to solve S-LQR. You are given a system (1) with

$$A = 0.1\begin{bmatrix} -1 & 1 & -2 \\ -3 & 3.2 & 3.5 \\ -5 & 6 & -7 \end{bmatrix} , \quad B = I \,,$$

   where $x_0 \sim \mathcal{N}(0, \Sigma_0)$ and $\Sigma_0 = I$. The average IH cost is defined as (2) as per Exercise 2.

   Your task is to implement the GD algorithm on MATLAB to compute the optimal LQR controller $K^\star$.

Task 1 Complete the passages in the pseudocode implementation of GD.

```
K0 = ???    (explain how to select an initial controller)


eta = ???    (explain, in words, how you would select a stepsize eta)


K = K0
while( ???                          )  (Insert a condition for stopping the algorithm)
    P_K = ???        (How to compute P_K?)


    Sigma_K = ???    (How to compute Sigma_K?)


    DeltaJ = ???     (How to compute the gradient DeltaJ)


    K = ???          (How to compute an updated value of K?)


end
```

Task 2 Complete the file "Ex14_3.m" to implement the GD algorithm. Verify that the GD converges to the optimal LQR controller.

4. In this exercise you will use projected GD to compute locally optimal distributed controllers. You are given a system (1) with

$$A = 0.2 \begin{bmatrix} -1 & 1 & -2 \\ -3 & 3.2 & 3.5 \\ -5 & 6 & -7 \end{bmatrix}, \quad B = I, \tag{4}$$

where $x_0 \sim \mathcal{N}(0, \Sigma_0)$ and $\Sigma_0 = I$. The average IH cost is defined as (2) as per Exercise 2 and Exercise 3. Notice that the definition of A has changed with respect to Exercise 3.

Task 1 You work for the company *ControlX* who is using a distributed controller

$$K_0 = \begin{bmatrix} -1.1189 & 0 & 0 \\ -0.6894 & 0 & -1.9114 \\ 0 & 1.7018 & 0 \end{bmatrix},$$

for their plant whose state-space model is given by (4). Letting $S = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, find a locally optimal controller $\hat{K} \in \text{Sparse}(S)$ using PGD starting from K_0. What PI can you achieve (see definition of PI in (3))?

Task 2 Your locally optimal controller from Task 1 is quite successful and improves the company's profits by $\sim 86\%$. Congratulations, you got a promotion!

A few months later your main competitor on the market, *iControl*, deploys a novel distributed controller $K_{\texttt{iControl}} \in \text{Sparse}(S)$ that significantly improves over your solution. How is it possible? Can you find a controller in the sparsity subspace $\text{Sparse}(S)$ that improves over $\hat{K}$? How much does it improve?