Multivariable Control (ME-422) - Exercise session 13
SOLUTIONS

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1. Consider the system

$$
\begin{aligned}
x_{k+1} &= x_k + w_k \qquad && w_k \sim WGN(0,1) \\
y_k &= x_k + v_k \qquad && v_k \sim WGN(0,2) \\
x_0 &\sim N(1,10)
\end{aligned}
$$

The measurements $y_0 = 2$ adn $y_1 = 3$ have been collected.

(a) Compute (by hand) $\hat{x}_{2|1}$ and $\Sigma_{2|1}$ using the Kalman predictor.

(b) Can you guarantee that $\Sigma_{k|k-1}$ converges as $k \to +\infty$? If yes, to which value does it converge?

Solution:

(a) Recall the KF formulae

$$
\begin{aligned}
&\hat{x}_{0|-1} = \mathrm{E}[x_0], \qquad \Sigma_{0|-1} = Var[x_0] \\
&\hat{x}_{k|k} = \hat{x}_{k|k-1} + \Sigma_{k|k-1} C^T \left(C\Sigma_{k|k-1}C^T + V\right)^{-1} \left(y_k - C\hat{x}_{k|k-1}\right) \\
&\Sigma_{k|k} = \Sigma_{k|k-1} - \Sigma_{k|k-1} C^T \left(C\Sigma_{k|k-1}C^T + V\right)^{-1} C\Sigma_{k|k-1} \\
&\hat{x}_{k+1|k} = A\hat{x}_{k|k} \\
&\Sigma_{k+1|k} = A\Sigma_{k|k}A^T + W
\end{aligned}
$$

For $A = 1$, $C = 1$, $W = 1$, $V = 2$ we have

- $k = 0$

$$
\begin{aligned}
\hat{x}_{0|0} &= 1 + 10\frac{1}{10+2}(2-1) = 1.833 \\
\Sigma_{0|0} &= 10 - 10^2\frac{1}{10+2} = 1.666 \\
\hat{x}_{1|0} &= \hat{x}_{0|0} = 1.833 \\
\Sigma_{1|0} &= \Sigma_{0|0} + 1 = 2.666
\end{aligned}
$$

- $k = 1$

$$
\begin{aligned}
\hat{x}_{1|1} &= 1.833 + 2.666\frac{1}{2.666+2}(3-1.833) = 2.5 \\
\Sigma_{1|1} &= 2.666 - 2.666^2\frac{1}{2.666+2} = 1.1429 \\
\hat{x}_{2|1} &= 2.5 \\
\Sigma_{2|1} &= 2.1429
\end{aligned}
$$

(b) One can set $W = B_q B_q^T$ for $B_q = 1$ and, since (A, B_q) is reachable and (A, C) is observable, $\Sigma_{k|k-1}$ converges to $\bar{\Sigma} > 0$ solving the ARE

$$\begin{aligned}
\bar{\Sigma} &= A\bar{\Sigma}A^T + W - A\bar{\Sigma}C^T \left[C\bar{\Sigma}C^T + V\right]^{-1} C\bar{\Sigma}A^T \\
\bar{\Sigma} &= \bar{\Sigma} + 1 - \frac{\bar{\Sigma}^2}{\bar{\Sigma} + 2} \\
\bar{\Sigma}^2 &- \bar{\Sigma} - 2 = 0 \\
\bar{\Sigma} &= \frac{1 \pm \sqrt{1+8}}{2} \quad \rightarrow \quad \bar{\Sigma} = 2.
\end{aligned}$$

2. The MATLAB code

```
% Generation of oscillator data

Ac=[0 1; -1 0];
Bc=[0;1];
Cc=[1 0];
Dc=0;
T=0.5;
sysC=ss(Ac,Bc,Cc,Dc);
sysD=c2d(sysC,T);

% noiseless input to the oscillator
Un=1*[ones(1,10) zeros(1,10) ones(1,10) zeros(1,11)];
% add process noise
sqW=0.1;
U=Un+sqW*randn(size(Un));

[Yn,Tsim,Xsim]=lsim(sysD,U);
% add measurement noise
sqV=0.2;
Ym=Yn+sqV*randn(size(Yn));

%% Plot of the output

figure(1);clf
%times=0:39;
PL=plot(Tsim,Ym);
NameArray = {'Color','LineStyle','LineWidth'};
ValueArray = {'blue','--',2};
set(PL,NameArray,ValueArray)

legend('measured outputs')
xlabel('{t}','FontSize',18)
ylabel('{y}','FontSize',18)
title('Oscillator data');
set(gca,'FontSize',13)

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Enter your code here for computing
% the time varying and time invariant
% Kalman predictors
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

available in Moodle under the name `problem2.m` generates noisy data from an oscillator driven by a square wave. The measurement noise is $WGN(0, 0.2^2)$ and the process noise is given by $B_c w_k$, where $w_k \sim WGN(0, 0.1^2)$. Assume that $x_0 \sim N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, I\right)$.

(a) Write the code for running a time-varying Kalman predictor. Show, in the same plot, the measured output and the estimate $\hat{y}_{k|k-1}$.

Hint: For debugging your code, set, in the simulator and the predictor, very small noise variances and, in the predictor, $\Sigma_{0|-1} = 10^{-6}I$. In this case, the real and predicted outputs must be almost identical.

(b) Write the code for implementing a steady-state Kalman predictor. Compare the results with those obtained in point (a).

Hint: Use the command `dare` for computing the steady-state covariance of the predictor. Note, however, that `dare` solves the IH-LQR problem.

(c) Let's now discuss the behavior of the predictor when the assumed variances are different from the true ones. Run simulations

- by dividing by 1000 the measurement noise covariance used in the Kalman predictor
- by multiplying by 1000 the measurement noise covariance used in the Kalman predictor.

Can you provide an intuitive explanation of the behavior of the predictor?

Solution: See the MATLAB file `Ex13_1.m` for the solutions.

3. Consider the system

$$\begin{aligned} x_{k+1} &= 0.5x_k + u_k + w_k \quad && w \sim WGN(0, 0.875) \\ y_k &= x_k + v_k && v \sim WGN(0, 1) \end{aligned}$$

where the noises are uncorrelated.

(a) Compute the infinite-horizon LQ (IH-LQ) control law with $Q = 0.875$ and $R = 1$

(b) Compute the corresponding closed-loop eigenvalue

(c) Compute the steady-state Kalman predictor

(d) Compute the overall output feedback LQG control law obtained by combining the previous results.

Hint: The discrete ARE (DARE) for IH-LQ control is given by

$$P = A^T PA + Q - A^T PB \left[B^T PB + R\right]^{-1} B^T PA \tag{1}$$

where the objective is to solve for $P = P^T > 0$ with a given choice of $Q = Q^T > 0$ and $R = R^T \geq 0$. Solve the parts (a)-(d) by hand. Then, verify your results by utilizing the MATLAB command `dare` to solve this equation for parts (a) and (c). However, while solving for part (c), be careful to take into account the fact that estimation problem is the dual of control problem, i.e., you have to replace A by A^T, B by C^T, Q by W, and R by V in (1).

Solution:

(a) For this system, the system matrices are

$$A = 0.5 \qquad B = 1 \qquad C = 1.$$

Also taking $Q = 0.875$ and $R = 1$, one obtains the DARE

$$P = 0.25P + 0.875 - 0.25\frac{P^2}{P+1}$$

which has a solution $P = 1$.
With this solution, the IH-LQ controller is $K = \left(B^T PB + R\right)^{-1} B^T PA = 0.25$.

(b) With $K = 0.25$, the closed-loop system matrix is $A - BK = 0.25$, yielding closed-loop eigenvalue of $\lambda_{cl} = 0.25$.

(c) DARE for Kalman Filtering is given by

$$\Sigma = 0.25\Sigma + 0.875 - 0.25\frac{\Sigma^2}{\Sigma + 1}$$

which is exactly the same equation as in part (a). Therefore, it has the same solution $\Sigma = 1$. The steady-state Kalman predictor gain is $L = A\Sigma C^T \left(C\Sigma C^T + V\right)^{-1} = 0.25$. The steady-state Kalman predictor is then

$$\hat{x}_{k+1|k} = 0.25\hat{x}_{k|k-1} + u_k + 0.25y_k. \tag{2}$$

(d) The overall output feedback LQR control law is defined, along with the steady-state Kalman predictor in (2), as

$$u_k = u_{k|k-1} = -K\hat{x}_{k|k-1} = -0.25\hat{x}_{k|k-1}.$$

4. **Sensor Fusion**
Consider a system of two sensors, each taking measurements of an unknown constant $\bar{x} \in \mathbb{R}$. Each measurement is noisy and modeled as follows

$$\begin{aligned} y^1 &= \bar{x} + v^1 \\ y^2 &= \bar{x} + v^2 \end{aligned}$$

where the measurement noises are correlated, i.e.,

$$v = \begin{bmatrix} v^1 \\ v^2 \end{bmatrix} \sim N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 0.1 \\ 0.1 & 1 \end{bmatrix}\right).$$

Asume that the measurements y_k, $k = 0, 1, \ldots, 200$ have been collected, where $y_k = \begin{bmatrix} y_k^1 \\ y_k^2 \end{bmatrix}$.

Estimate $\bar{x}$ in the framework of Kalman filtering. Adapt the code developed in problem 2 for developing a time-varying KF, simulating the data for $\bar{x} = 5$, and plotting $\hat{x}_{k+1|k}$.

Hint: For generating a Gaussian random vector $v \in \mathbb{R}^2$ with $Var[v] = V$, use the MATLAB command `w=sqrtm(V)*randn(2,1)`.

Solution: To use the KF framework, consider the following model

$$\begin{aligned} x_{k+1} &= x_k \\ y_k &= Cx_k + v_k \end{aligned}$$

where $C = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $x_0 \sim N(1, 1)$ ($\mathrm{E}[x_0] = 1$ and $Var[x_0] = 1$ are arbitrary choices). The solution file is `Ex13_2.m`.