

Multivariable Control (ME-422) - Exercise session 10

SOLUTIONS

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1. Steady-State Control of a Scalar System

Consider the system

$$x_{k+1} = ax_k + bu_k,$$

with the cost

$$J = \frac{1}{2} \sum_{k=0}^{\infty} (qx_k^2 + ru_k^2).$$

Assume that the pair (a, b) is reachable and the pair $(a, \sqrt{q})$ is observable.

(a) Let $u_k = -\bar{K}x_k$ be the LQR law. Show that the closed-loop system is

$$x_{k+1} = a^{cl}x_k, \quad a^{cl} = \frac{a}{1 + (b^2/r)\bar{P}} \quad (1)$$

where $\bar{P} > 0$ is a solution (if it exists) of the ARE

$$\frac{\Lambda}{q}\bar{P}^2 + (1 - \Lambda)\bar{P} - \frac{\Lambda r}{b^2} = 0 \quad \Lambda = \frac{b^2 q}{(1 - a^2)r}. \quad (2)$$

(b) Equation (2) has two solutions given by

$$\bar{P} = \frac{q}{2\Lambda} \left[\pm \sqrt{(1 - \Lambda)^2 + \frac{4\Lambda}{(1 - a^2)}} - (1 - \Lambda) \right].$$

We now consider two cases.

- i. Assume that $|a| < 1$ (the open-loop system is asymptotically stable). Show that, for all $q \geq 0$, the closed-loop system (1) is asymptotically stable as well.
- ii. Assume that $|a| > 1$ (the open-loop system is unstable).
 - Show that if $|a| \gg 1$, the closed-loop system (1) is asymptotically stable.
 - When $|a| > 1$, it is not easy to prove that $|a^{cl}| < 1$. It is possible, however, to compute a^{cl} as a function of a and Λ (you have probably done it already in the previous point). Assuming that $\frac{b^2 q}{r} = 1$, using MATLAB, plot the values of a^{cl} as a function of $a \in [-20, 20]$.

Solution:

(a) The optimal control minimizing J is the constant feedback $u_k = -\bar{K}x_k$, where, as seen in the lectures, the gain is

$$\bar{K} = \frac{ab\bar{P}}{b^2\bar{P} + r} \quad (3)$$

and $\bar{P}$ is the unique positive (definite) root of the algebraic Riccati equation

$$\bar{P} = a^2\bar{P} - \frac{a^2b^2\bar{P}^2}{b^2\bar{P} + r} + q.$$

Under the influence of the control, the closed-loop system is

$$a^{cl} = a - b\bar{K} = \frac{a}{1 + (b^2/r)\bar{P}}$$

The ARE can be written

$$b^2\bar{P}^2 + [(1 - a^2)r - b^2q]\bar{P} - qr = 0$$

If we define the auxiliary variable

$$\Lambda = \frac{b^2q}{(1 - a^2)r},$$

this becomes

$$\frac{\Lambda}{q}\bar{P}^2 + (1 - \Lambda)\bar{P} - \frac{\Lambda r}{b^2} = 0$$

- (b) i. If $|a| < 1$, one has $(1 - a^2) > 0$ and $\Lambda > 0$. In this case, the unique non-negative solution to (2) is

$$\bar{P} = \frac{q}{2\Lambda} \left[\sqrt{(1 - \Lambda)^2 + \frac{4\Lambda}{(1 - a^2)}} - (1 - \Lambda) \right],$$

and the steady-state gain is given by (3).

In the scalar case, observability of $(a, \sqrt{q})$ is equivalent to $q \neq 0$, and plant reachability is equivalent to $b \neq 0$. Therefore, the observability and reachability conditions imply $\Lambda > 0$ and hence $\bar{P} > 0$. Then, according to (1)

$$|a^{cl}| < |a|,$$

so that the closed-loop system is stable.

Since $|a| < 1$, if $q = 0$, the system is still detectable (the unobservable mode is stable). If $q = 0$, then $\Lambda = 0$, but according to (1), a^{cl} is still stable. Note that in this case, $\bar{P} = 0$ (i.e., positive *semidefinite*).

- ii. If $|a| > 1$, one has $(1 - a^2) < 0$ and $\Lambda < 0$. In this case, the unique non-negative solution to (2) is

$$\bar{P} = \frac{-q}{2\Lambda} \left[\sqrt{(1 - \Lambda)^2 + \frac{4\Lambda}{(1 - a^2)}} + (1 - \Lambda) \right].$$

Again, reachability and observability imply that Λ is strictly negative, so that according to the equation above, $\bar{P} > 0$.

According to (1), if $\Lambda < 0$, we still have $|a^{cl}| < |a|$, but it is not easy to prove that $|a^{cl}| < 1$. Note, however, that

$$a^{cl} = \frac{a}{1 - \frac{1-a^2}{2} \left[\sqrt{(1 - \Lambda)^2 + \frac{4\Lambda}{(1-a^2)}} + (1 - \Lambda) \right]},$$

so that if $|a| \gg 1$, then $\Lambda \simeq 0$ and

$$a^{cl} \simeq \frac{1}{a}.$$

This is certainly stable.

Note that a^{cl} does not depend on b , q , and r individually, but only on the quantity b^2q/r (which is also true if a is stable). If $|a| > 1$, then detectability of $(a, \sqrt{q})$ is equivalent to $q \neq 0$, which is necessary for a^{cl} to be stable.

- See the MATLAB file `Ex10.1.m` for the solution.

2. Inverted Pendulum on Cart

In this exercise, you will design an LQR controller for the cart-pendulum system for offset-free tracking of reference linear position. To this end, the controller will have an internal integrator

state. This is the same system that you have seen in the previous exercise session. All the files that you will need for the simulation are provided on Moodle.

Download the file *Inverted+pendulum+2018-4+no+solutions.zip* and unzip it. Consider the inverted pendulum described in the file *Inverted+Pendulum+description.pdf* and already used in the previous exercise session. See Figure 1 for the schematic description of the system.

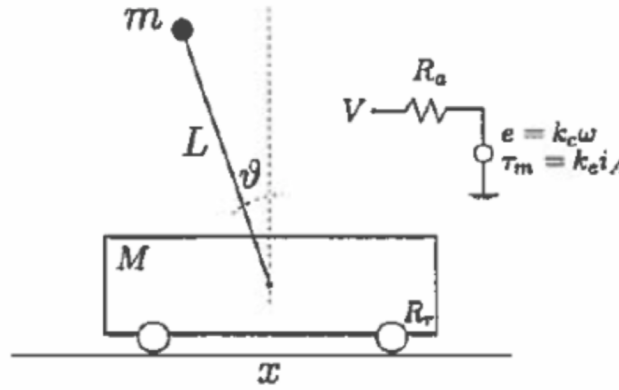


Figure 1: Cart-pendulum system.

By discretizing, with sampling period $T = 0.1s$, the model obtained through linearization about the equilibrium corresponding to the vertical position, one obtains

$$x^+ = \underbrace{\begin{bmatrix} 1 & 0.0824 & -0.0087 & -0.0003 \\ 0 & 0.6692 & -0.1652 & -0.0087 \\ 0 & 0.0178 & 1.0582 & 0.1019 \\ 0 & 0.3367 & 1.1652 & 1.0582 \end{bmatrix}}_{A_d} x + \underbrace{\begin{bmatrix} 0.0009 \\ 0.0165 \\ -0.0009 \\ -0.0168 \end{bmatrix}}_{B_d} u$$

- (a) Design an LQR controller on the extended system (including the integrator on the linear position variable). For setting the values of Q and R , use the normalization approach, knowing that, typically

$$|x_1| \leq \sqrt{5} \quad |x_2| \leq \sqrt{0.05} \quad |x_3| \leq \sqrt{5} \quad |x_4| \leq \sqrt{0.05} \quad |x_5| \leq \sqrt{0.05} \quad |u| \leq \sqrt{500}.$$

Set the free parameters in Q and R equal to 0.5, to start with.

Hint: Use the MATLAB function `K=dlqr(A,B,Q,R)`.

- (b) Open *LQ_integrator_pendulum_anim.slx* and run a simulation. When using the pulse input, the tracking of the linear position is unsatisfactory. Just by changing a single entry in Q or R , design a new controller for improving the tracking.
- (c) Try now to improve the tracking performance (compared to the one obtained in part (a)) by designing an LQ controller with a prescribed degree of stability, as seen in the lectures.

Solution: See the MATLAB file `Ex10.2.m`.