Lecture 8

Optimal control

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Linear Quadratic (LQ) optimal control

LQ control over a finite horizon (FH)

System under control

$$x^+ = Ax + Bu \quad x(k) \in \mathbb{R}^n \quad u(k) \in \mathbb{R}^m \tag{1}$$

$$x(0) = x_0 \tag{2}$$

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Optimal control cost

$$J = \left\{ \sum_{k=0}^{N-1} x^T(k)Qx(k) + u^T(k)Ru(k) \right\} + x^T(N)Sx(N) \tag{3}$$

where

$$Q \in \mathbb{R}^{n\times n}, Q = Q^T \geq 0 \qquad S \in \mathbb{R}^{n\times n}, S = S^T \geq 0$$

$$R \in \mathbb{R}^{m\times m}, R = R^T > 0 \qquad N \in \mathbb{N}, N \geq 1$$

Goal of optimal control

Compute $u(0)$, $u(1)$, ..., $u(N-1)$ that minimises J under the constraints (1) at times $0, 1, \ldots, N-1$ and constraint (2) (the state x_0 is measured)

Terminology :

- The problem is called LQ because it refers to a Linear system and a Quadratic cost
- N: control horizon. It is finite.

Minor remark

In (3), $x^T(0)Qx(0) \in \mathbb{R}$ is a term that could be omitted. It is kept for notational simplicity

Linear Quadratic (LQ) optimal control

Analysis of the cost

$$x^T(k)Qx(k) + u^T(k)Ru(k) \qquad \rightarrow \text{stage cost}$$
$$x(N)^T Sx(N) \qquad \rightarrow \text{terminal cost}$$

- $x^T(k)Qx(k)$ penalises "big states". Same for the terminal cost.

Linear Quadratic (LQ) optimal control

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- $u^T(k)Ru(k)$ penalises "big inputs", which are actuator-unfriendly
  - It is in conflict with the terms above: for steering rapidly $x(k) \rightarrow 0$ one usually need a large amount of "control energy".

Linear Quadratic (LQ) optimal control

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Design parameters

Control horizon N and matrix weights Q, S, R.

e.g. : $Q >> R$, $S >> R$ means that we want to steer the state to zero quickly and we do not care much about how big the control variables are.

Linear Quadratic (LQ) optimal control

Properties of Q, S, R

- We assume they are symmetric matrices, without loss of generality. Indeed, for a generic $T \in \mathbb{R}^{n\times n}$, the quadratic form $x^T Tx$ can be written as

$$x^T Tx = \underbrace{x^T \frac{T+T^T}{2} x}_{(a)} + x^T \frac{T-T^T}{2} x = (a) + x^T \frac{T}{2} x - \underbrace{x^T \frac{T^T}{2} x}_{scalar}$$

$$= (a) + x^T \frac{T}{2} x - x^T \frac{T}{2} x = (a)$$

  (a) is called the symmetric part of T

- $Q \geq 0$, $S \geq 0$ but $R > 0$: zero penalty is ok on states but not on the input (see later why ...)
  $\hookrightarrow$ They guarantee that $J \geq 0$ (key property of a meaningful "cost")

Linear Quadratic (LQ) optimal control

Solution to the FH-LQ problem

Theorem (solution to FH-LQ)

There is a unique control law

$$u(k) = -K(k)x(k) \quad k = 0, \ldots, N-1$$

minimising J, where the control gains are computed by the following algorithm

- Set $P(N) = S$
- For $k = N-1, N-2, \ldots, 0$

$$K(k) = \left[R + B^T P(k+1) B\right]^{-1} B^T P(k+1) A \tag{R1}$$

$$P(k) = Q + A^T P(k+1) A - K(k)^T \left[R + B^T P(k+1) B\right] K(k) \tag{R2}$$

Moreover $J^* = \min\limits_{u(0),\ldots,u(N-1)} J = x^T(0)P(0)x(0)$.

Linear Quadratic (LQ) optimal control

Solution to the FH-LQ problem

- Substituting $K(k)$ in $(R2)$ one gets the recursive update

$$P(k) = Q + A^T P(k+1)A - A^T P(k+1)B \left[R + B^T P(k+1)B\right]^{-1} B^T P(k+1)A$$

Known as Difference Riccati Equation (DRE)

Linear Quadratic (LQ) optimal control

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Known as Difference Riccati Equation (DRE)

- Backward iterations
- S, Q, R symmetric $\Rightarrow$ $P(k)$ symmetric $k = 0, \dots, N-1$
- $S, Q, R \geq 0$ and $\left[R + B^T P(k+1)B\right]$ invertible
  $\Rightarrow \quad P(k) \geq 0 \quad k = 0, \dots, N-1$

Linear Quadratic (LQ) optimal control

Solution to the FH-LQ problem

- The gain $K(k)$
  - is time-varying $\rightarrow$ the closed-loop system is linear time-varying
  - is defined only for $k = 0, \ldots, N-1 \rightarrow$ "stability" of the closed-loop system has no meaning
  - can be precomputed at time $k = 0$ and independently of x_0
  - $u(k) = -K(k)x(k)$ is a state-feedback regulator

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  - can be precomputed at time $k = 0$ and independently of x_0
  - $u(k) = -K(k)x(k)$ is a state-feedback regulator
- The assumption that $R > 0$ guarantees that $R + B^T P(k+1)B$ is invertible

Review : minimization of quadratic forms

Consider the quadratic form

$$F(x) = x^T H x + x^T g + g^T x, \quad H = H^T > 0$$

Linear Quadratic (LQ) optimal control

Proof of the Theorem

Review : minimization of quadratic forms

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The minimum can be computed setting $\frac{dF}{dx} = 0$. Recall that

$$\frac{\partial}{\partial a} a^T B c = c^T B^T \quad \frac{\partial}{\partial c} a^T B c = a^T B.$$

Linear Quadratic (LQ) optimal control

Proof of the Theorem

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Then

$$\frac{dF}{dx} = 0 \Leftrightarrow x^T H + x^T H^T + g^T + g^T = 2x^T H + 2g^T = 0$$

and we have

$$x = -H^{-1} g$$

Linear Quadratic (LQ) optimal control

Proof of the theorem

We use a dynamic programming argument. Define the "cost-to-go"

$$V(k) = \min_{u(k),u(k+1),\dots,u(N-1)} \left\{ \sum_{i=k}^{N-1} x^T(i)Qx(i) + u^T(i)Ru(i) \right\} + x^T(N)Sx(N)$$

which is the queue of the cost from k to N.

For $k = N$,

$$V(N) = x^T(N)P(N)x(N) = x^T(N)Sx(N)$$

For $k = N - 1$,

$$
\begin{aligned}
V(N-1) &= \min_{u(N-1)} \underbrace{x^T(N-1)Qx(N-1) + u^T(N-1)Ru(N-1)}_{(a)} + V(N) \\
&= \min_{u(N-1)} (a) + (Ax(N-1) + Bu(N-1))^T P(N)(Ax(N-1) \\
&\quad + Bu(N-1)) \\
&= \min_{u(N-1)} x^T(N-1)\left[Q + A^T P(N) A\right] x(N-1) + u^T(N-1) \\
&\quad \left[R + B^T P(N) B\right] u(N-1) + 2x^T(N-1)A^T P(N) Bu(N-1)
\end{aligned}
$$

Linear Quadratic (LQ) optimal control

Proof of the theorem

We must minimize a quadratic form in the variable $u(N-1)$, treating $x(N-1)$ as a given vector of parameters (it is indeed measured).

$$\frac{\partial V(N-1)}{\partial u(N-1)} = 2u^T(N-1)\left[R + B^T P(N) B\right] + 2x^T(N-1)A^T P(N) B = 0$$

This gives

$$u(N-1) = \underbrace{-\left[R + B^T P(N) B\right]^{-1} B^T P(N) A}_{K(N-1)} x(N-1) =$$

The next goal is to express $V(N-1)$ as a quadratic form.

Linear Quadratic (LQ) optimal control

Proof of the theorem

Substituting $u(N-1)$ into $V(N-1)$ we have

$$
\begin{aligned}
V(N-1) =& x^T(N-1)\underbrace{\left[Q + A^T P(N) A\right]}_{(b)} x(N-1) + \\
& x^T(N-1)\underbrace{A^T P(N) B\left[R + B^T P(N) B\right]^{-1} B^T P(N)}_{(c)} A x(N-1) \\
& -2x^T(N-1)\underbrace{A^T P(N) B\left[R + B^T P(N) B\right]^{-1} B^T P(N)}_{\text{identical to } (c)} A x(N-1)
\end{aligned}
$$

Defining $P(N-1) = (b) - (c)$ (which is $(R2)$ in the Theorem) one has

$$V(N-1) = x^T(N-1) P(N-1) x(N-1)$$

which has the same structure of $V(N)$.

Linear Quadratic (LQ) optimal control
Proof

At time $k = N - 2$, one can replicate the above steps by realizing that

$$V(N-2) = \min_{u(N-2)} x^T(N-2)Qx(N-2) + u^T(N-2)Ru(N-2) + V(N-1) \quad (*)$$

Formula $(*)$ is the Bellman iteration and is the core of the dynamic programming procedure.
Proceeding iteratively for $k = N-2, N-1, \ldots, 0$ one obtains the algorithm in the statement of the Theorem. ■

Linear Quadratic (LQ) optimal control

Example

Problem

DT system under control

$$x(k+1) = 0.3679x(k) + 0.6321u(k), \quad x(0) = 1$$

Determine the optimal control law to minimize the following performance index :

$$J = x(10)^2 + \sum_{k=0}^{9} \left(x^2(k) + u^2(k)\right)$$

Note that in this example $S = 1$, $Q = 1$, and $R = 1$. Also, determine the minimum value of J.

Linear Quadratic (LQ) optimal control

Example - Solution

Using the Riccati equation, we obtain $P(k)$ as follows:

$$\begin{aligned} P(k) &= 1 + (0.3679)^2 P(k+1) \\ &\quad - 0.3679 P(k+1) 0.6321^2 (1 + 0.6321^2 P(k+1))^{-1} P(k+1) 0.3679 \end{aligned}$$

which can be simplified to

$$P(k) = 1 + 0.1354 P(k+1)[1 + 0.3996 P(k+1)]^{-1}$$

The boundary condition for $P(k)$ is

$$P(N) = P(10) = S = 1$$

Linear Quadratic (LQ) optimal control

Example - Solution

We now compute $P(k)$ backward from $k = 9$ to $k = 0$:

$$
\begin{aligned}
P(9) &= 1.0967 \\
P(8) &= 1.1032 \\
P(7) &= 1.1036 \\
P(6) &= 1.1037 \\
P(k) &= 1.1037, \quad k = 5, 4, 3, 2, 1, 0
\end{aligned}
$$

Notice that the values of $P(k)$ rapidly approach the steady-state value.
The steady-state value P_{ss} can be obtained from

$$P_{ss} = 1 + 0.1354 P_{ss}(1 + 0.3996 P_{ss})^{-1}$$

or

$$0.3996 P_{ss}^2 + 0.4650 P_{ss} - 1 = 0$$

Solving this last equation for P_{ss}, we have

$$P_{ss} = 1.1037 \quad \text{or} \quad -2.2674$$

Linear Quadratic (LQ) optimal control

Example - Solution

Since $P(k)$ must be positive, we find the steady-state value for $P(k)$ to be 1.1037.

The feedback gain $K(k)$ can be computed as

$$K(k) = \left[1 + 0.6321^2 P(k+1)\right]^{-1} 0.6321 P(k+1)(0.3679)$$

By substituting the values of $P(k)$ we have obtained, we get

$$\begin{aligned}
K(10) &= 0 \\
K(9) &= 0.1662 \\
K(8) &= 0.1773 \\
K(7) &= 0.1781 \\
K(6) &= K(5) = \cdots = K(0) = 0.1781
\end{aligned}$$

The optimal control law is given by

$$u(k) = -K(k)x(k)$$

Linear Quadratic (LQ) optimal control

Example - Solution

Since

$$x(k+1) = 0.3679x(k) + 0.6321u(k) = 10.3679 - 0.6321K(k)x(k)$$

we obtain

$$\begin{aligned} x(1) &= [0.3679 - 0.6321K(0)]x(0) \\ &= (0.3679 - 0.6321 \times 0.1781) \times 1 = 0.2553 \\ x(2) &= (0.3679 - 0.6321 \times 0.1781) \times 0.2553 = 0.0652 \\ x(3) &= (0.3679 - 0.6321 \times 0.1781) \times 0.0652 = 0.0166 \\ x(4) &= (0.3679 - 0.6321 \times 0.1781) \times 0.0166 = 0.00424 \end{aligned}$$

The values of $x(k)$ for $k = 5, 6, \ldots, 10$ approach zero rapidly.

Linear Quadratic (LQ) optimal control

Example - Solution

The optimal control sequence $u(k)$ is now obtained as follows:

$$
\begin{aligned}
u(0) &= -K(0)x(0) = -0.1781 \times 1 = -0.1781 \\
u(1) &= -K(1)x(1) = -0.1781 \times 0.2553 = -0.0455 \\
u(2) &= -K(2)x(2) = -0.1781 \times 0.0652 = -0.0116 \\
u(3) &= -K(3)x(3) = -0.1781 \times 0.0166 = -0.00296 \\
u(4) &= -K(4)x(4) = -0.1781 \times 0.00424 = -0.000756 \\
u(k) &\simeq 0, \quad k = 5, 6, \ldots, 10
\end{aligned}
$$

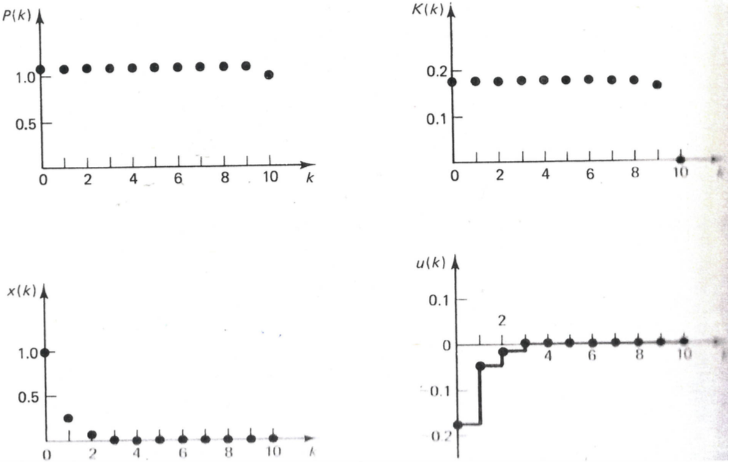
Finally, the minimum value of the performance index J can be obtained as

$$
J^* = x(0)P(0)x(0) = (1 \times 1.1037 \times 1) = 1.1037
$$

Linear Quadratic (LQ) optimal control

Example - Solution

The values of $P(k)$, $K(k)$, $x(k)$, and $u(k)$ are shown in the figure below. Notice that the values of $P(k)$ and $K(k)$ are constant except for the final few stages.



Infinite-horizon LQ optimal control

Infinite-Horizon (IH) LQ optimal control

Main drawback of FH-OC : $u(k)$ is defined only for $k = 0, \ldots, N-1$

Idea : consider the case $N \to +\infty$.

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$$J = \sum_{k=0}^{+\infty} x^T(k)Qx(k) + u^T(k)Ru(k), \quad Q = Q^T \geq 0, R = R^T > 0$$

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Goal

Find the sequence $u(0), u(1), \ldots$ minimizing J

Infinite-Horizon (IH) LQ optimal control

Remark

There could be no sequence $u(0), u(1), \ldots$ for which $J < +\infty$!

Infinite-Horizon (IH) LQ optimal control

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Example

$$x^+ = 2x + 0 \cdot u, x_0 = 1 \quad Q = R = 1 \underbrace{\rightarrow}_{\text{for } u(\cdot)=0} J = \sum_{k=0}^{+\infty} 2^k Q 2^k = +\infty$$

If $u(\cdot) \neq 0$, any nonzero input sample $u(k)$ gives a positive contribution $\rightarrow$ the cost is still $+\infty$

Remark

The terminal cost $x^+(\infty)Sx(\infty)$ has no meaning

Solution to IH-LQ

Theorem

Assume (A, B) is reachable. Then

1. For any initial condition $P(N) = S = S^T \geq 0$, the matrices $P(k)$, $k = N-1, N-2, \ldots, 0, -1, -2, \ldots$ converge to a symmetric matrix $\bar{P}$ which is the unique positive-semidefinite solution of the Algebraic Riccati Equation (ARE)

$$\bar{P} = A^T \bar{P} A + Q - A^T \bar{P} B (R + B^T \bar{P} B)^{-1} B^T \bar{P} A$$

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$$\bar{K} = (R + B^T \bar{P} B)^{-1} B^T \bar{P} A$$

3. The optimal cost (corresponding to the optimal control law) is

$$J^* = x(0)^T \bar{P} x(0)$$

Solution to IH-LQ

Remarks

- Part 1 establishes the link between FH-LQ and IH-LQ
  - $\bar{P}$ is independent of the initial condition $P(N) = S$
  - (A, B) reachable and $Q \geq 0$, $R > 0 \Rightarrow$ the ARE always has a solution $\bar{P} \geq 0$
    - non obvious, as the ARE is a nonlinear matrix equation in $\bar{P}$

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- Part 2 : the IH-LQ control law is time-invariant $\rightarrow$ the closed-loop system dynamics is

$$x^+ = (A - B\bar{K})x$$

  - The assumption (A, B) reachable can be weakened into (A, B) stabilisable (meaning the unreachable eigenvalues have modulus < 1)

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Terminology: $u(k) = -\bar{K}x(k)$ is called the LQ Regulator (LQR)

Open problems: is $\bar{P} > 0$? Is $A - B\bar{K}$ Schur?

Tools for studying if $\bar{P} > 0$

Factorisation :

$$Q = C_1^T C_1 \quad C_1 \in \mathbb{R}^{n_c \times n}$$

Not unique. Example $Q = 1 \rightarrow C_1 = 1$ or $C_1 = -1$

Lemma

Let C_1 and C_2 be factors such that $Q = C_1^T C_1 = C_2^T C_2$.
Then

$$(A, C_1) \text{ observable } \Leftrightarrow (A, C_2) \text{ observable}$$

Proof of the Lemma

If (A, C_1) observable, then $\bigcap_{J=0}^{n-1} \mathrm{Ker}(C_1 A^J) = \{0\}$, where $\mathrm{Ker}(F)$ is the null space of F.
Assume by contradiction that (A, C_2) is unobservable. Then, $\bigcap_{J=0}^{n-1} \mathrm{Ker}(C_2 A^J), \neq \{0\}$, hence implying that $\exists x_0 \in \bigcap_{J=0}^{n-1} \mathrm{Ker}(C_2 A^J)$, $x_0 \neq 0$, such that $C_2 A^k x_0 = 0$, for $k = 0, 1, \ldots$.
Therefore

$$x_0^T (A^k)^T C_2^T C_2 A^k x_0 = x_0^T (A^k)^T Q (A^k) x_0 = x_0^T (A^k)^T C_1^T C_1 A^k x_0 = 0$$

which implies $C_1 A^k x_0 = 0$, $k = 0, 1, \ldots$. But this contradicts the observability of (A, C_1). ■

Why observability of (A, C_1) is important ?

- Consider $J = \sum_{k=0}^{+\infty} x(k)^T Qx(k) + u^T(k)Ru(k)$, $x^+ = Ax + Bu$ and evaluate the cost of the control law $u(k) = 0$. One has

$$J = \sum_{k=0}^{+\infty} x_0^T (A^T)^k QA^k x_0 \qquad (**)$$

Since $Q \geq 0$, one might have $x_0 \neq 0$ producing $J = 0$. But if (A, C_1) is observable, this cannot happen, i.e.

$$J = 0 \Rightarrow x_0 = 0 \qquad (***)$$

Why observability of (A, C_1) is important ?

- Consider $J = \sum_{k=0}^{+\infty} x(k)^T Q x(k) + u^T(k) R u(k)$, $x^+ = Ax + Bu$ and evaluate the cost of the control law $u(k) = 0$. One has

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$$J = 0 \Rightarrow x_0 = 0 \qquad (***)$$

Remarks

If $Q = C_1^T C_1 > 0$, one can show that (A, C_1) is always observable

Frequent case : $Q = \text{diag}(q_1, \ldots, q_n), \quad q_i > 0$

Why observability of (A, C_1) is important ?

Lemma

If $Q = C_1^T C_1$ and (A, C_1) observable, then the solution $\bar{P} \geq 0$ of the ARE is positive definite.

Why observability of (A, C_1) is important ?

Lemma

If $Q = C_1^T C_1$ and (A, C_1) observable, then the solution $\bar{P} \geq 0$ of the ARE is positive definite.

Proof

By contradiction assume $\bar{P}$ positive semidefinite.
Since $J^* = x(0)^T \bar{P} x(0)$, there is $x(0) \neq 0$ such that $J^* = 0$.
Since $R > 0$, $J^* = 0 \Rightarrow u(0) = u(1) = \cdots = 0$.
Therefore, the cost J^* coincides with J on $(**)$. But, as shown in $(***)$ this would imply $x_0 = 0$, which is a contradiction.

Stabilizing LQR

Open-loop system and LQR

$$x^+ = Ax + Bu$$
$$u = -\bar{K}x$$

Closed-loop (CL)

$$x^+ = (A - B\bar{K})x$$

$\bar{K}$ = time-invariant matrix gain produced by ∞-horizon LQ problem

Theorem (CL stability of LQR)

Let C_1 be a matrix verifying $Q = C_1^T C_1$. If

- (a) (A, B) is reachable
- (b) (A, C_1) is observable

then $A - B\bar{K}$ is Schur stable.

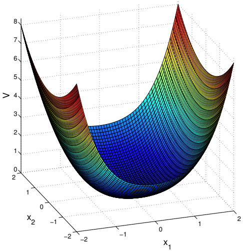
Before providing the proof, interlude: Lyapunov stability theory

Lyapunov stability theory

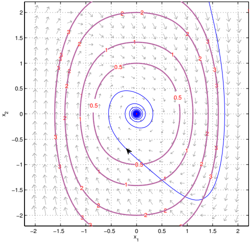
- We focus on **stability of the origin** for the LTI system $x^+ = Ax$
- Idea: if an energy-like function of the state decreases to zero, the origin is stable.

Lyapunov stability theory

Energy $V(x)$



(x_1,x_2)-plane



- $V(x)$ is a measure of the distance of x from the origin.
  - If $V(x)$ can only decrease, then $\bar{x} = 0$ should be stable.
- Next: make statements more rigorous!

Energy forward difference

$$x^+ = Ax$$

Consider a quadratic energy-like function: $V(x) = x^T P x$, where $P \in \mathbb{R}^{n \times n}$ is symmetric and positive definite

- Compute $\Delta V(x) = V(x(k+1)) - V(x(k))$

$$\Delta V(x) = x^T A^T P A x - x^T P x = x^T (A^T P A - P) x$$

- We are sure that $\Delta V(x) \leq 0$ if

$$A^T P A - P \leq 0$$

Lyapunov theorems

Theorem 1: stability

The LTI system $x^+ = Ax$ is stable, if and only if there is $P > 0$ such that $A^T PA - P \leq 0$

Theorem 2 (AS)

For the LTI system $x^+ = Ax$, the following statements are equivalent

(a) the system is AS

(b) for *an arbitrary matrix* $Q > 0$, there is a matrix $P^T = P > 0$ solving the Lyapunov equation

$$A^T PA - P = -Q$$

Lyapunov theorems

Terminology

- $V(x) = x^T P x$ is a candidate Lyapunov function
- If $V(x)$ verifies one of the two theorems, it is a Lyapunov function

Remark

- $A^T PA - P = -Q$ is a system of linear equations in the elements of P, for a given Q

Proof of the stability theorem

Recall the statement

Theorem (CL stability of LQR)

Let C_1 be a matrix verifying $Q = C_1^T C_1$. If

a. (A, B) is reachable
b. (A, C_1) is observable

then $A - B\bar{K}$ is Schur stable.

For simplicity, we will discuss the proof under the simplifying assumption that $Q > 0$ (instead of $Q \geq 0$)

Proof

$(a)+(b)$ guarantees that the unique solution of the ARE is $\bar{P}>0$. Let $V(x)=x^T\bar{P}x$ be a candidate Lyapunov function.
Forward difference :

$$\begin{aligned}\Delta V(x) = (x^+)^T\bar{P}x^+ - x\bar{P}x &= x^T(A-B\bar{K})^T\bar{P}(A-B\bar{K})x - x^T\bar{P}x \\ &= x^T \underbrace{\left[(A-B\bar{K})^T\bar{P}(A-B\bar{K}) - \bar{P}\right]}_{(*)} x \qquad (1)\end{aligned}$$

We want to show that $(*)$ is negative definite. To this purpose note that

$$\begin{aligned}(\bar{K}^TB^T)\bar{P}(B\bar{K})-(\bar{K}^TB^T)\bar{P}A+\bar{K}^TR\bar{K} &= \bar{K}^T(B^T\bar{P}B+R)\bar{K}-\bar{K}^TB^T\bar{P}A \\ &= \bar{K}^T\left[(B^T\bar{P}B+R)\underbrace{(B^T\bar{P}B+R)^{-1}B^T\bar{P}}_{\bar{K}}A - B^T\bar{P}A)\right] = 0\end{aligned}$$

Proof

Adding $(\bar{K}^T B^T)\bar{P}(B\bar{K}) - (\bar{K}^T B^T)\bar{P}A + \bar{K}^T R\bar{K}$ to the ARE we obtain

$$\underbrace{\bar{P} = A^T\bar{P}A + Q - A^T\bar{P}B\bar{K}}_{\text{ARE}} + \bar{K}^T B^T \bar{P}B\bar{K} - \bar{K}^T B^T \bar{P}A + \bar{K}^T R\bar{K}$$

that is

$$\bar{P} = [A - B\bar{K}]^T \bar{P}[A - B\bar{K}] + Q + \bar{K}^T R\bar{K}$$

This gives

$$(*) = [A - B\bar{K}]^T \bar{P}[A - B\bar{K}] - \bar{P} = -(Q + \bar{K}^T R\bar{K}) \qquad (2)$$

Proof

Substituting (2) in (1) we have

$$\Delta V(x) = -x^T(Q + \bar{K}^T R \bar{K})x$$

Since $Q > 0$ and $R > 0$, then $Q + \bar{K}^T R \bar{K} > 0$ and $\Delta V(x) < 0$ unless $x = 0$. From Lyapunov stability theory, the closed-loop system is asymptotically stable.
If, instead, $Q \geq 0$, then $Q + \bar{K}^T R \bar{K} \geq 0$ and one has to show that there are no state trajectories $\bar{x}(k)$ giving $\Delta V(\bar{x}(k)) = 0$ at all times, except for the trivial one $\bar{x}(\cdot) = 0$. This can be done by exploiting the assumption (b).

Remarks on LQR

- The stability theorem holds true even under the weaker assumptions that
  - a') (A, B) is stabilizable
  - b') (A, C_1) is detectable

- (A, B) is stabilizable if any unreachable eigenvalue λ of A verifies $|\lambda| < 1$
- (A, C_1) is detectable if any unobservable eigenvalue λ of A verifies $|\lambda| < 1$

Remarks on LQR

- If the system (A, B, C) is observable, one can show that (b) can be always verified by choosing $Q = C^T \bar{Q} C$, where $\bar{Q} = \bar{Q}^T \in \mathbb{R}^{P \times P}$, $\quad \bar{Q} > 0$ is a design parameter.
  The associated cost is

$$J = \sum_{k=0}^{+\infty} x^T(k) C^T \bar{Q} C x(k) + u^T(k) R u(k)$$
$$= \sum_{k=0}^{+\infty} y^T(k) \bar{Q} y(k) + u^T(k) R u(k)$$

  where the output y has been penalized instead of the state.

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  where the output y has been penalized instead of the state.
- LQR can be applied to multi-input systems $\rightarrow$ Interesting for the multivariable case!
- If $x(k)$ is not measured, one can replace it with an estimate $\bar{x}(k)$ provided by a state observer (see later in the course).

LQR with prescribed stability degree

Problem of standard LQR: some $\lambda \in \text{Spec}(A - B\bar{K})$ could be close to the boundary of stability region, that is $|\lambda| \simeq 1$.

LQR with prescribed stability degree

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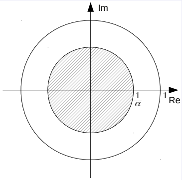
Discounted LQR

$$J_\alpha = \sum_{k=0}^{+\infty} \left(x^T(k)Qx(k) + u^T(k)Ru(k)\right) \alpha^{2k}, \quad \alpha > 1$$

Theorem (discounted LQR)

Let $\bar{K}_\alpha$ be the LQR gain associated to J_α. then

$$\text{Spec}(A - B\bar{K}_\alpha) \subset \frac{1}{\alpha} B(0,1) \qquad (\blacksquare)$$



Proof of the theorem

We have

$$J_\alpha = \sum_{k=0}^{+\infty} \hat{x}^T(k) Q \hat{x}(k) + \hat{u}^T(k) R \hat{u}(k) \qquad (*)$$

with $\hat{x}(k) = \alpha^k x(k)$ and $\hat{u}(k) = \alpha^k u(k)$.
Multiplying $x^+ = Ax + Bu$ by α^{k+1}, we have

$$\hat{x}^+ = \underbrace{\alpha A}_{\hat{A}} \hat{x} + \underbrace{\alpha B}_{\hat{B}} \hat{u} = \hat{A}\hat{x} + \hat{B}\hat{u} \qquad (**)$$

Then $(*)$ and $(**)$ define a standard LQ problem and the associated LQR guarantees that $\hat{x} \to 0$ as $k \to \infty$. Hence, for $x^+ = (A - B\bar{K})x$, one has that $x \to 0$ at least as fast as $\left(\frac{1}{\alpha}\right)^k$. In view of the relations between modes and eigenvalues, we have (■).

LQR with prescribed stability degree of stability

Algorithm

- Define $\hat{A} = \alpha A$ and $\hat{B} = \alpha B$
- Compute $\hat{K}$ from the standard LQR problem with weights Q and R
- Use $u(k) = -\hat{K}x(k)$

Design of weights Q and R

- No golden rule - a few common criteria in the sequel
- Often some trial-and-error is required for achieving satisfactory performances

Design of weights Q and R

Modal analysis

- If A has very different eigenvalues, as in the figure below

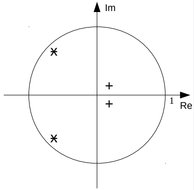


Figure: Eigenvalues providing satisfactory modes $(+)$ and highly oscillating modes $(*)$

a sensible goal is to move the critical eigenvalues $(*)$ and focus less on the remaining ones.

Idea : In J assign more weights to selected modes. How to do it ?

Design of weights Q and R

Assume A has distinct eigenvalues $\lambda_1, \lambda_2, \ldots,$ implying that A can be diagonalized by a non-singular matrix T containing eigenvectors as columns. Change of coordinates : $\bar{x} = Tx$,
$\bar{A} = TAT^{-1} = \text{diag}(\lambda_1, \ldots, \lambda_n)$, $\bar{B} = TB$

$$\bar{x}^+ = \bar{A}\bar{x} + \bar{B}u \qquad (\star)$$

Remark

$\bar{x}_i$ is associated with an eigenvalue λ_i.

Formulation of LQR for $(\star)$

$$\bar{J} = \sum_{k=0}^{+\infty} \bar{x}^T(k)\bar{Q}\bar{x}(k) + u^T(k)Ru(k)$$

Design of weights Q and R

- Choose $\bar{Q} = \text{diag}(\bar{q}_1, \dots, \bar{q}_n)$ and weight more the "undesired" eigenvalues
- We have
$$\bar{x}^T(k)\bar{Q}\bar{x}(k) = x^T(k)T^T\bar{Q}Tx(k)$$
Set $Q = T^T\bar{Q}T$ in the LQR problem for the original system.
- The above method can be generalized to multiple eigenvalues through the use of Jordan forms (not in this class).

Design of weights Q and R: the normalization approach

$$J = \sum_{k=0}^{+\infty} q_1 x_1^2(k) + \dots q_n x_n^2(k) + r_1 u_1^2(k) + \dots r_m u_m^2(k)$$

Problem: each variable might be measured in different units $\rightarrow$ weights are scale-dependent

Idea

Assume one knows that $\|u_j\| \leq u_{j,max}$ $j = 1, \dots, m$ and $\|x_i\| \leq x_{i,max}$, $i = 1, \dots, n$.
Define $q_i = \frac{\tilde{q}_i}{x_{i,max}^2}$ and $r_j = \frac{\tilde{r}_j}{u_{j,max}^2}$. The cost becomes

$$J = \sum_{k=0}^{+\infty} \frac{\tilde{q}_1}{x_{1,max}^2} x_1^2(k) + \dots \frac{\tilde{q}_n}{x_{n,max}^2} x_n^2(k) + \frac{\tilde{r}_1}{u_{1,max}^2} u_1^2(k) + \dots \frac{\tilde{r}_m}{u_{m,max}^2} u_m^2(k)$$

and $\tilde{q}_i$, $\tilde{r}_i$ can be chosen in the interval $(0, 1)$, independently of the measurement units