Lecture 4

Discretisation of Continuous Time (CT) systems

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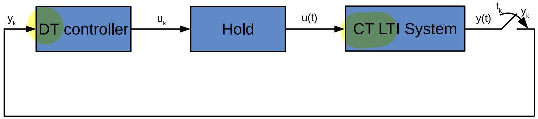
Motivation: digital control of CT systems



- $y(t)$: plant measurements
- y_k: input to controller

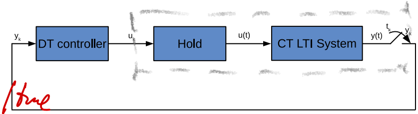
- Plant dynamics

$$
\begin{aligned}
\dot{x} &= Ax + Bu & \qquad x(t) &\in \mathbb{R}^n \\
y &= Cx + Du & u(t) &\in \mathbb{R}^p \\
x(0) &= x_0 & y(t) &\in \mathbb{R}^m
\end{aligned}
$$

Model of sample and hold

- System CT output $y(t)$ sampled at $\{t_k, k \in \mathbb{N}\}$
  - $y_k = y(t_k)$, $T_k \triangleq t_{k+1} - t_k$

    ↳ sampling interval / time

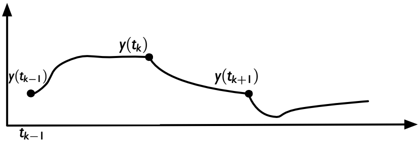


- Hold

$$u(t) = u_k \quad t \in [t_k, t_{k+1})$$

- **Problem**: obtain DT models representing the cascade Hold + (CT system) + Sampler

- **Next**: popular discretisation methods
  - exact
  - approximate

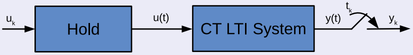


Exact discretisation

Goal: compute the discrete-time dynamics for x_k

LTI system

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$



- Set $x_k = x(t_k)$, $y_k = y(t_k)$ etc.
- Recall the Lagrange formula for the above system with $x(t_k) = x_k$

$$x(t) = e^{A(t-t_k)}x_k + \underbrace{\int_{t_k}^{t} e^{A(t-\tau)}Bu(\tau)d\tau}_{(b)}$$

State transition operator e^{As}: pushes x_k ahead by s seconds

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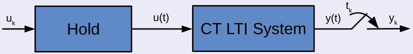

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t_k t

$$\psi(z) = \begin{bmatrix} \psi_{11}(z) & \dots & \psi_{1n}(z) \\ \vdots & & \\ \psi_{n1}(z) & \dots & \psi_{nn}(z) \end{bmatrix}$$

Constant-input transmission operator $\Gamma(s) \triangleq \int_0^s e^{Az}dz$

- $(b) = \Gamma(t - t_k)Bu_k$ if $u(.) = u_k$ on $[t_k, t]$.

  Proof: $(b) = \int_{t_k}^{t} e^{A(t-\tau)}d\tau Bu_k = -\int_{t-t_k}^{0} e^{Az}dzBu_k = \int_0^{t-t_k} e^{Az}dzBu_k$, where we have set $z = t - \tau$.

- $\Gamma(t - t_k)B$ pushes u_k ahead by $t - t_k$ seconds

Exact discretisation

- Sample at times $t_0 = 0, t_1, t_2, \dots$ and set $T_k = t_{k+1} - t_k$.
- We have

$$x_{k+1} = x(t_{k+1}) = e^{A(t_{k+1}-t_k)}x(t_k) + \int_{t_k}^{t_{k+1}} e^{A(t_{k+1}-s)}Bu(s)\,ds$$

$$= \underbrace{e^{AT_k}}_{\hat{A}_k} x_k + \underbrace{\Gamma(T_k)B}_{\hat{B}_k} u_k \quad (1)$$

$$y_k = \hat{C}x_k + \hat{D}u_k \quad \text{for} \quad \hat{C} = C, \hat{D} = D \quad (2)$$

- (1) - (2) : linear time-varying system

Under uniform sampling (i.e, $T_k = T,\ k = 0, 1, \dots$)

$$\hat{A} = e^{AT}, \quad \hat{B} = \Gamma(T)B \rightarrow \text{ invariant system}$$

$$\begin{cases} x^+ = \hat{A}x + \hat{B}u \\ y = \hat{C}x + \hat{D}u \end{cases}$$

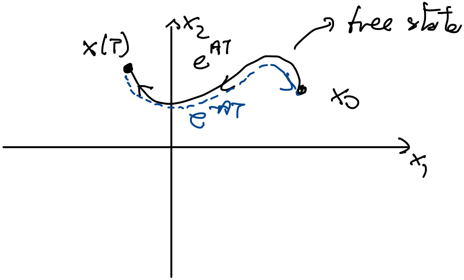
Eigenvalues under exact sampling (1/2)

$$\hat{A} = e^{AT}$$

Properties

One can show that

- (a) $\lambda \in \text{Spec}(A) \Rightarrow z = e^{\lambda T} \in \text{Spec}(\hat{A})$ $\qquad a = 0 \rightarrow \hat{a} = e^{aT} = 1$
- (b) $\det(\hat{A}) \neq 0$ always (even if $\det(A) = 0$)
  Moreover $(e^{AT})^{-1} = e^{-AT} = e^{A(-T)}$ (inverse = "backward in time")



$$e^{-AT} x(T) = \underbrace{e^{-AT} e^{AT}}_{I} x(0) = x(0)$$

Eigenvalues under exact sampling (1/2)

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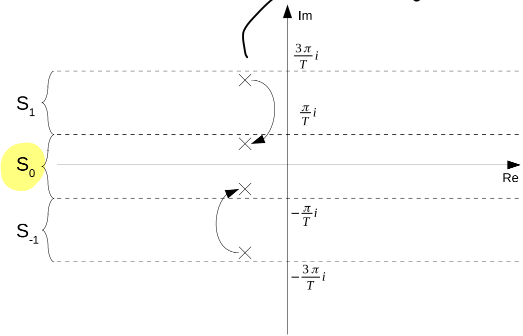
- Implications of (b) :
  reachability and controllability coincide for DT systems obtained through exact sampling
- Implications of (a) :
  $\lambda_1 \neq \lambda_2$ does not imply that $z_1 \neq z_2$
  Check : take $\lambda_2 = \lambda_1 + j\frac{2\pi}{T}N$, $N = \pm 1, \pm 2, \pm 3, \ldots$ $N \neq 0$

  We have $z_2 = \underbrace{e^{\lambda_1 T}}_{z_1} \underbrace{e^{j\frac{2\pi}{\cancel{T}}N\cancel{T}}}_{1}$

Interpretation in the complex plane

eigenvalues of the CT system



- S_i = Strips of width $2\pi/T$
- Due to sampling, all strips "collapse" into S_0
  Sampling $\Rightarrow$ loss of information: can not reconstruct eigenvalues of the CT system from those of the DT one.

Eigenvalues under exact sampling (2/2)

Map of regions in the λ-plane (continuous time) into regions of the z-plane (discrete-time).

two c.c. eigenvalues with real part < 0

$z = e^{\lambda T}$

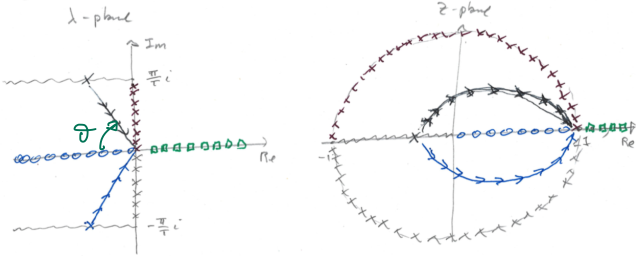
$\frac{s^2}{\omega_n^2} + \frac{2\xi}{\omega_n} s + 1$

$\omega_n > 0$ natural frequency

$\xi \in (0, 1]$ damping

$\xi = \cos(\vartheta)$

λ-plane	z-plane
0	1
real negative	$[0, 1)$
real positive	$(1, +\infty)$
constant imaginary part	$(-1, 0]$
constant damping	log spirals



- Remark: asymptotically stable eigenvalues λ (i.e. $\text{Re}(\lambda) < 0$) are mapped into asymptotically stable eigenvalue z (i.e. $|z| < 1$).

Exact sampling: conclusions

- Pros
  - x_k and y_k represent $x(t_k)$ and $y(t_k)$ with no approximations
  - Stability, AS and instability are preserved
- Cons
  - $\hat{A} = e^{AT} = I + AT + \frac{(AT)^2}{2} + \dots$
  - can be difficult to compute for large systems !
    In Matlab, `hA=expm(A*T)`
  - does not preserve the pattern of zeros in A

Example

$$A = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix}, \quad T = 0.1$$

$$\rightarrow \quad \hat{A} = e^{AT} = \begin{bmatrix} 0.905 & 0.0905 & 0.0045 \\ 0.0045 & 0.905 & 0.0905 \\ 0.0905 & 0.0045 & 0.905 \end{bmatrix}$$

Exact sampling: conclusions

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Generalisations

What happens if $u(t)$ or $x(t)$ are affected by delays ? $\rightarrow$ see the class "Networked Control Systems"

Approximate discretisation methods

- Goal: avoid the computation of e^{AT} and preserve the structure of A

CT LTI System

$$\dot{x} = Ax + Bu \tag{1}$$

$$y = Cx + Du \tag{2}$$

Assumption

Uniform sampling period T

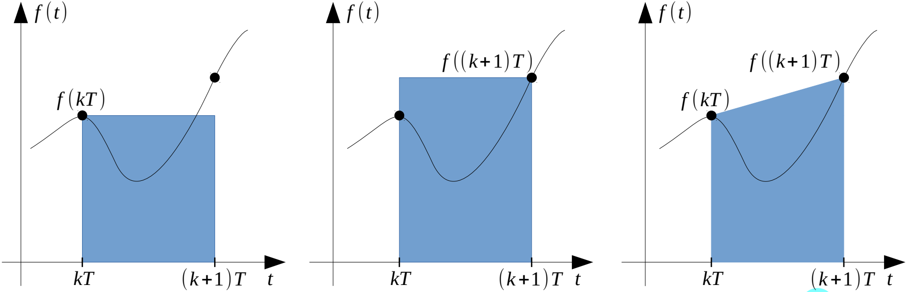
- Integrating (1) on $[kT, (k+1)T]$ and setting $x_k = x(kT)$ gives

$$x_{k+1} - x_k = A \int_{kT}^{(k+1)T} x(t)\, dt + B \int_{kT}^{(k+1)T} u(t)\, dt$$

Approximate discretisation methods

- To make $u(kT)$ and $x(kT)$ appear in the right hand side, we use the following numerical integration scheme, written for a generic function $f(t) : \mathbb{R} \to \mathbb{R}^n$

$$\int_{kT}^{(k+1)T} f(t)\,dt \simeq [(1-\alpha)f(kT) + \alpha f((k+1)T)]\,T, \quad 0 \leq \alpha \leq 1$$



Forward Euler (α=0) Backward Euler (α=1) Tustin (α=0.5)

DT Models

$\alpha = 0$	:	Forward Euler (FE)
$\alpha = 1$	:	Backward Euler (BE)
$\alpha = 0.5$	:	Trapezoidal method or Tustin method

In all cases, loss of information. Is stability preserved ?

Resulting DT system

$$x_{k+1} - x_k = A\left[(1-\alpha)x_k + \alpha x_{k+1}\right]T + BT\left[(1-\alpha)u_k + \alpha u_{k+1}\right]$$

- $\alpha = 0$ (FE)

$$x_{k+1} = (AT + I)x_k + BTu_k$$

- $\alpha = 1$ (BE)

$$x_{k+1} = ATx_{k+1} + x_k + BTu_{k+1}$$

- $\alpha = 0.5$ (Tustin)

$$x_{k+1} = (\frac{1}{2}AT + I)x_k + \frac{1}{2}ATx_{k+1} + BT\frac{1}{2}\left[u_k + u_{k+1}\right]$$

1h24 d

Analysis of FE ($\alpha = 0$)

Remark

FE provides a causal DT system with $\hat{A} = AT + I$, $\hat{B} = BT$

Lemma (properties of FE)

(a) The off-diagonal zero entries of A and $AT + I$ are in the same positions

$\mathrm{Re}(\lambda) < 0$ $|z| < 1$

(b) if A is Hurwitz stable, then $AT + I$ is Schur stable if and only if

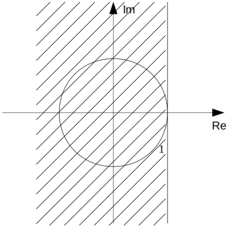
$$T < T^* = \min_{i=1,\dots,n} \frac{-2\,\mathrm{Re}(\lambda_i)}{|\lambda_i|^2} \tag{1}$$

eigenvalue of the A matrix

where λ_i are the eigenvalues of A.

Remark

If λ is an eigenvalue of A, then $\lambda T + 1$ is an eigenvalue of $AT + I$. The function $\lambda \mapsto \lambda T + 1$ maps the set $\{\lambda \in \mathbb{C} : \text{Re}(\lambda) < 0\}$ in the shaded region below



Remark

$$e^{AT} = I + AT + \cancel{\frac{(AT)^2}{2}} + \cancel{\cdots}$$

Therefore FE can be seen as a first order Taylor approximation of e^{AT}

Effect of discretization on reachability and observability

- Discretisation $\Rightarrow$ loss of information
- We expect that sampling may impair reachability/observability...

CT LTI System

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned} \qquad (*)$$

Theorem

Assume that $(*)$ is reachable ~~controllable~~ and observable. The DT system obtained through exact discretisation is reachable ~~controllable~~ and observable if and only if, for any pair of eigenvalues λ_i, λ_j of A

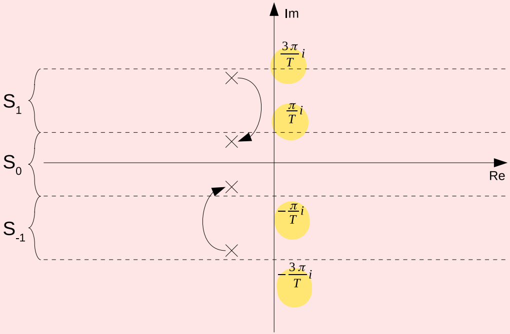
$$\mathrm{Re}(\lambda_i) = \mathrm{Re}(\lambda_j) \Rightarrow \mathrm{Im}(\lambda_i - \lambda_j) \neq \frac{2n\pi}{T} \quad n = \pm 1, \pm 2, \ldots \quad n \neq 0$$

where T is the sampling period.

Effect of discretization on reachability and observability

Remarks

- Recall the relations between eigenvalues under exact sampling



- The CT system has real eigenvalues only $\Rightarrow$ no loss of reachability/observability
- Always possible to avoid losses of reachability/observability by choosing $T > 0$ small enough

Example

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \qquad (**)$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

- Check at home that $(**)$ is reachable and observable (the tests are the same for CT and DT systems)
- System eigenvalues $\lambda_1 = j, \quad \lambda_2 = -j$

DT system (sampling time $T > 0$)

$$\begin{bmatrix} x_1^+ \\ x_2^+ \end{bmatrix} = \underbrace{\begin{bmatrix} \cos(T) & \sin(T) \\ -\sin(T) & \cos(T) \end{bmatrix}}_{\hat{A}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 1-\cos(T) \\ \sin(T) \end{bmatrix}}_{\hat{B}} u$$

$$y = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{\hat{C}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Reachability of the DT system

$$M_r = \left[\begin{array}{cc} \hat{B} & \hat{A}\hat{B} \end{array}\right] = \left[\begin{array}{cc} 1-\cos(T) & \cos(T)+1-2\cos^2(T) \\ \sin(T) & -\sin(T)+2\cos(T)\sin(T) \end{array}\right]$$

$$\text{Rank}(M_r) = 2 \quad \Leftrightarrow \quad T \neq n\pi \quad n = 1, 2, 3 \ldots$$

$\sin(n\pi) = 0$ zero second row for $T = n\pi$

- By applying the Theorem
  $\text{Re}(\lambda_1) = \text{Re}(\lambda_2) = 0$. Then, the system is reachable if and only if

$$\underbrace{\text{Im}(\lambda_1 - \lambda_2)}_{\text{Im}(2j)=2} \neq \frac{2n\pi}{T} \quad \Leftrightarrow \quad T \neq n\pi$$

Conclusions

- Time-discretization $\Rightarrow$ loss of information
- Exact sampling : best possible choice but it does not
  - preserve structure
  - apply to nonlinear systems

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$\rightarrow$ What about the simultaneous presence of sampling and time delays?
Master course "Networked Control Systems"