Lecture 3

Observability and duality

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Section 1

Observability

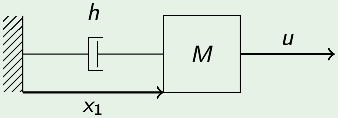
Observability

$$x^+ = Ax + Bu \tag{1}$$
$$y = Cx + Du \tag{2}$$
$$x(0) = x_0 \tag{3}$$

$x \in \mathbb{R}^n, u \in \mathbb{R}^m, y \in \mathbb{R}^p$

Is it possible to reconstruct the initial state $x(0)$ from a finite number of *free* output samples?

Observability - Example



Damping coef $h = 1$
$M = 1$
Output y =velocity of M

Setting $x_2 = \dot{x}_1$

$$\begin{cases} \dot{x}_1 = x_2 \\ M\dot{x}_2 = -hx_2 + u \\ y = x_2 = \text{velocity} \end{cases}$$

Discretization:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{x(k+1) - x(k)}{T}, \; T = 0.1 \implies \begin{cases} x_1^+ = x_1 + 0.1x_2 \\ x_2^+ = 0.9x_2 + 0.1u \\ y = x_2 \end{cases}$$

- The output does not depend on x_1. Impossible to reconstruct $x_1(0)$ by observing $y(k), k \geq 0$.
- Does anything change if $y = x_1$?

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Observability

Definition

$$x^+ = Ax + Bu \tag{4}$$
$$y = Cx + Du \tag{5}$$

$x \in \mathbb{R}^n, u \in \mathbb{R}^m, y \in \mathbb{R}^p$

Definition

A state $\tilde{x} \neq 0$ is **unobservable** if, $\forall k > 0$, the free output

$$\tilde{y}_f(k) = C\phi(k, 0, \tilde{x}, 0) + D0$$

verifies $\tilde{y}_f(k) = 0$.
A system without unobservable states is **observable**.

Observability

Definition

Remarks

- $x(0) = 0 \implies y_f(k) = 0, \ \forall k \geq 0$. Hence, if $x(0) = \tilde{x} \neq 0$ is unobservable, it cannot be distinguished from $x(0) = 0$ by only observing the free output → one cannot reconstruct unambiguously the initial state from the free output
- Observability: property of the pair (A, C) only

Problem

Difficult to check if a system is observable using the definition (infinitely many $\tilde{x}$ should be tested)

Observability test

Definition

The **observability matrix** is defined as

$$M_o = \begin{bmatrix} C^T & A^T C^T & \cdots & (A^T)^{n-1} C^T \end{bmatrix} \in \mathbb{R}^{n \times pn}$$

Theorem

1. $\tilde{x} \neq 0$ *is unobservable only if it belongs to*

   $$X_{no} = Ker(M_o^T) \rightarrow \overbrace{\begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}}^{M_o^T} \tilde{x} = 0$$

   where $Ker(M_o^T)$ *is the nullspace of* M_o^T

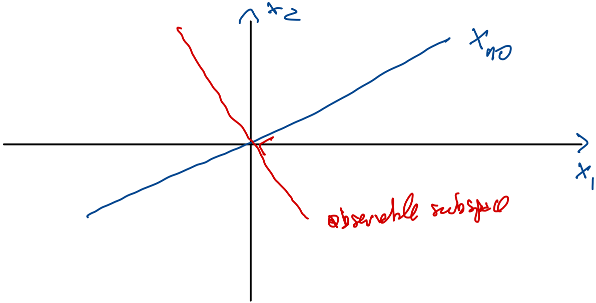
2. *If the free output* $\tilde{y}_f(\cdot)$ *generated by* $\tilde{x}$ *is non-identically null, then* $\exists \tilde{k} \cancel{\leq n} < n$ *such that* $\tilde{y}_f(\tilde{k}) \neq 0$

3. *The system is observable if and only if* $rank(M_o) = n$

Observability test

Remarks

- point 3 $\Longrightarrow$ For LTI systems, observability is a finitely determined property
- The orthogonal subspace $X_{no}^{\perp}$ is termed the « observable subspace »
- Careful: $x \notin X_{no}$ is not enough to say that x is observable, as it might contain a component lying in X_{no}



Observability test

Proof

The unobservable initial state $x(0) = \tilde{x}$ must verify the equations

$$\begin{aligned} y_f(0) &= C\tilde{x} = 0 \\ y_f(1) &= CA\tilde{x} = 0 \\ &\vdots \\ y_f(k) &= CA^k\tilde{x} = 0 \end{aligned} \implies \overbrace{\begin{bmatrix} C \\ CA \\ \vdots \\ CA^k \end{bmatrix}}^{M_o^T} \tilde{x} = 0$$

From the theorem of Cayley-Hamilton (*c.f.* proof of reachability test), the rows of CA^n are linear combinations of the rows of $CA^i, i = 0, 1, \ldots, n-1$. Hence,

$$y_f(k) = CA^k\tilde{x} = 0, k = 0, \ldots, n-1 \implies y_f(\hat{k}) = CA^{\hat{k}}\tilde{x} = 0, \hat{k} \geq n$$

↳ Negation

$\exists \hat{k} \geq n$ s.t $y_f(\hat{k}) \neq 0 \Rightarrow \exists k \in \{0, \ldots n-1\}$ s.t $y_f(k) \neq 0$

Observability test

The previous implication shows that

$$\tilde{x} \in \mathrm{Ker}(M_o^T) \implies \tilde{x} \text{ unobservable}$$

It also shows point (2). Point (3) follows from the fact that

$$\mathrm{rank}(M_o) = n \Leftrightarrow \mathrm{Ker}(M_o^T) = \{0\}$$

$\square$

Unobservable systems

If the system is unobservable, the unobservable part can be isolated.

Theorem

Let $n_o = rank(M_o) \geq 1$. There is a suitable and non-unique change of state coordinates

$$\hat{x}(k) = T_o x(k), \ \det(T_o) \neq 0$$

such that $x^+ = Ax$, $y = Cx$ is equivalent to

$$\hat{x}^+ = \hat{A}\hat{x}, \ y = \hat{C}\hat{x}$$

where

$$\hat{A} = \begin{bmatrix} \hat{A}_a & 0 \\ \hat{A}_{ba} & \hat{A}_b \end{bmatrix} \qquad \hat{x} = \begin{bmatrix} \hat{x}_a \\ \hat{x}_b \end{bmatrix} \qquad \hat{A}_a \in \mathbb{R}^{n_o \times n_o}$$

$$\hat{C} = \begin{bmatrix} \hat{C}_a & 0 \end{bmatrix} \qquad \hat{C}_a \in \mathbb{R}^{p \times n_o}$$

$$rank\left(\begin{bmatrix} \hat{C}_a^T & \hat{A}_a^T \hat{C}_a^T & (\hat{A}_a^T)^{n_o - 1} \hat{C}_a^T \end{bmatrix}\right) = n_o \tag{6}$$

Terminology: $(\hat{A}, \hat{C})$ is called the « observability form » of (A, C)

Unobservable systems

Remarks

The zero blocks in $(\hat{A}, \hat{C})$ reveal the unobservable part. Setting $\hat{x} = \begin{bmatrix} \hat{x}_a^T & \hat{x}_b^T \end{bmatrix}^T$, $\hat{x}_a \in \mathbb{R}^{n_o}$ we have

$$\hat{x}_a^+ = \hat{A}_a \hat{x}_a \tag{7}$$

$$\hat{x}_b^+ = \hat{A}_{ba}\hat{x}_a + \hat{A}_b \hat{x}_b \tag{8}$$

$$y = \hat{C}_a \hat{x}_a \tag{9}$$

Free outputs generated by $\hat{x}_a(0) = 0$ and arbitrary $\hat{x}_b(0)$ are null.
Eq. (7) is the observable part: since $(\hat{A}_a, \hat{C}_a)$ is observable (c.f. eq. (6)), $\hat{x}_a(0)$ can be always reconstructed observing the output for, at most, n_0 steps.

Terminology

The eigenvalues of $\hat{A}_a$ are termed « observable ».
Those of $\hat{A}_b$, « unobservable ».
The same applies to the corresponding modes.

How to build T_o?

- Build $M_o = \begin{bmatrix} C^T & A^TC^T & \cdots & (A^T)^{n-1}C^T \end{bmatrix}$ and let $v_1, \ldots, v_{n-n_o}$ be linearly independent column vectors such that

$$M_o^T v_i = 0$$

(v_i ~~forms~~ form a basis for X_{no}).

- Build $T_o^{-1} = \begin{bmatrix} z_1 & \cdots & z_{n_o} & | & v_1 & \cdots & v_{n-n_o} \end{bmatrix}$ where z_i are arbitrary vectors guaranteeing that $\det\left(T_o^{-1}\right) \neq 0$.

$$x = T_o^{-1}\begin{bmatrix} \hat{x}_a \\ \hat{x}_b \end{bmatrix} = \begin{bmatrix} z_1 \cdots z_{n_o} \mid v_1 \cdots v_{n-n_o} \end{bmatrix}\begin{bmatrix} \hat{x}_a \\ \hat{x}_b \end{bmatrix}$$

Example (ctd)

$$x^{+} = \begin{bmatrix} 1 & 0.1 \\ 0 & 0.9 \end{bmatrix} x$$
$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} x$$

Remark: swapping x_1 and x_2 gives the matrices $\hat{A}, \hat{c} \ldots$.

Blind computations

$$M_o = \begin{bmatrix} C^T & A^T C^T \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0.9 \end{bmatrix}. \quad n_o = \text{rank}(M_o) = 1$$

$$M_o^T v_1 = 0 \implies \begin{bmatrix} 0 & 1 \\ 0 & 0.9 \end{bmatrix} v_1 = 0 \to \text{pick } v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$T_o^{-1} = \begin{bmatrix} z_1 & | & v_1 \end{bmatrix}$$

Pick $z_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and obtain $T_o^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. One has $T_o = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $\hat{A} = T_o A T_o^{-1} = \begin{bmatrix} 0.9 & 0 \\ 0.1 & 1 \end{bmatrix}$, $\hat{C} = C T_o^{-1} = \begin{bmatrix} 1 & 0 \end{bmatrix}$

Section 2

Principle of duality

Principle of duality

$$\Sigma : \begin{cases} x^+ & = Ax + Bu \\ y & = Cx + Du \end{cases} \quad \Sigma^D : \begin{cases} x^+ & = A^T x + C^T u \\ y & = B^T x + Du \end{cases}$$

Note the correspondences $A \to A^T$, $B \to C^T$, $C \to B^T$.

Definition

Σ^D is the dual of the system Σ

Main property

Σ is reachable [resp. observable] iff Σ^D is observable [resp. reachable]

Check: Assume Σ reachable: $M_r = \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}$. Observability matrix of Σ^D: $M_o = \begin{bmatrix} (B^T)^T & (A^T)^T(B^T)^T & \cdots & \left((A^T)^T\right)^{n-1}(B^T)^T \end{bmatrix}$ $= \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}$. We will see more on duality when discussing controllers and state estimators.

Section 3

PBH test

PBH (Popov, Belevitch, Hautus) Lemmas

$$x^+ = Ax + Bu$$
$$y = Cx + Du$$

- Let $\lambda \in \mathrm{Spec}(A)$. Is λ a reachable/observable eigenvalue?

Lemma

The eigenvalue λ is

- **reachable** if and only if

$$\mathrm{rank}[\lambda I - A \mid B] = n \tag{10}$$

- **observable** if and only if

$$\mathrm{rank} \begin{bmatrix} \lambda I - A \\ C \end{bmatrix} = n \tag{11}$$

PBH (Popov, Belevitch, Hautus) lemma

Proof of "(10)$\Rightarrow$ λ is reachable"

By contradiction, assume (10) true but λ is unreachable. We know that $\exists T$: $\det(T) \neq 0$ such that

$$TAT^{-1} = \hat{A} = \begin{bmatrix} \hat{A}_a & \hat{A}_{ab} \\ 0 & \hat{A}_b \end{bmatrix}, \quad \hat{B} = \begin{bmatrix} \hat{B}_a \\ 0 \end{bmatrix}$$

$\hat{A}_b$ identifies the unreachable part and thus $\lambda \in \mathrm{Spec}(\hat{A}_b)$.
Since $\hat{A}_b$ and $\hat{A}_b^T$ have the same eigenvalues, this means that

$$\exists \hat{x}_b \neq 0 : \; \hat{A}_b^T \hat{x}_b = \lambda \hat{x}_b \tag{12}$$

Let

$$x = T^T \begin{bmatrix} 0 \\ \hat{x}_b \end{bmatrix}, \text{ and hence } x^T = \begin{bmatrix} 0 & \hat{x}_b^T \end{bmatrix} T$$

Therefore

$$x^T \begin{bmatrix} \lambda I - A & B \end{bmatrix} = x^T T^{-1} \begin{bmatrix} \lambda T - TA\overbrace{T^{-1}T}^{I} & TB \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \hat{x}_b^T \end{bmatrix} \underbrace{TT^{-1}}_{I} \left[\lambda T - \begin{bmatrix} \hat{A}_a & \hat{A}_{ab} \\ 0 & \hat{A}_b \end{bmatrix} T \,\middle|\, \begin{bmatrix} \hat{B}_a \\ 0 \end{bmatrix} \right]$$

PBH (Popov, Belevitch, Hautus) lemma

Proof cont.

$$
\begin{aligned}
&= \left[\lambda[0 \ \hat{x}_b^T] T - [0 \ \hat{x}_b^T] \begin{bmatrix} \hat{A}_a & \hat{A}_{ab} \\ 0 & \hat{A}_b \end{bmatrix} T \,\middle|\, [0 \ \hat{x}_b^T] \begin{bmatrix} \hat{B}_1 \\ 0 \end{bmatrix} \right] \\
&= \left[[0 \quad \lambda \hat{x}_b^T] T - [0 \quad \hat{x}_b^T \hat{A}_b] T \,\middle|\, \underbrace{0}_{=0 \cdot T} \right] \\
&= \left[0 \quad \hat{x}_b^T [\lambda I - \hat{A}_b] \mid 0 \right] T \\
&= \left[0 \quad \left([\lambda I - \hat{A}_b^T] \hat{x}_b \right)^T \mid 0 \right] T \\
&= 0
\end{aligned}
$$

where the last equality follows from (12). Since $x^T[\lambda I - A \mid B] = 0$, we have shown that $\text{rank}[\lambda I - A \mid B] < n$, which contradicts (10). □

For the observability test (11), one can ~~give~~ provide a similar proof.

Example (ctd)

$$x^+ = \begin{bmatrix} 1 & 10 \\ 0 & -9 \end{bmatrix} x$$
$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} x$$

Is the eigenvalue $\lambda = 1$ unobservable?

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 10 \\ 0 & -9 \end{bmatrix}$$

$$\text{rank} \begin{bmatrix} \lambda I - A \\ C \end{bmatrix} = \text{rank} \begin{bmatrix} 0 & -10 \\ 0 & 10 \\ 0 & 1 \end{bmatrix} = 1 < 2 \implies \text{YES}$$

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