

Lecture 2

Stability, reachability and controllability

Giancarlo Ferrari Trecate¹

¹Dependable Control and Decision Group
École Polytechnique Fédérale de Lausanne (EPFL), Switzerland
giancarlo.ferraritrecate@epfl.ch

Section 1

Modes and stability of LTI systems

Modes and stability of LTI systems

$$\begin{cases} x^+ = Ax + Bu \\ x(0) = x_0 \end{cases} \quad (1)$$

$(\bar{x}, \bar{u})$ equilibrium

- Recall: stability of the equilibrium state $\bar{x}$.

Definition (Lyapunov stability)

The equilibrium state $\bar{x}$ is

- stable if $\forall \epsilon > 0 \exists \delta > 0 : \|\tilde{x}_0 - \bar{x}\| \leq \delta \Rightarrow \|\tilde{x}(k) - \bar{x}\| < \epsilon, \forall k \geq 0$
- (globally) asymptotically stable (AS) if it is stable and attractive, i.e.,

$$\lim_{k \rightarrow \infty} \|\tilde{x}(k) - \bar{x}\| = 0, \forall \tilde{x}_0 \in \mathbb{R}^n$$

- unstable if not stable

Modes and stability of LTI Systems

- Key quantity to analyse: the **error**

$$e(k) = \tilde{x}(k) - \bar{x}$$

Proposition

Set $e_0 = e(0) = \tilde{x}(0) - \bar{x}$. The error verifies

$$\begin{aligned} e^+ &= Ae \\ e(0) &= e_0 \end{aligned} \tag{2}$$

Modes and stability of LTI systems

Proof

$$\tilde{x}(k) = \phi(k, 0, \tilde{x}_0, \bar{u})$$

$$\bar{x}(k) = \phi(k, 0, \bar{x}, \bar{u})$$

Note that $e(k) = \alpha \tilde{x}(k) + \beta \bar{x}(k)$ with $\alpha = 1$ and $\beta = -1$. From the superposition principle,

$$e(k) = \tilde{x}(k) - \bar{x}(k) = \phi(k, 0, \tilde{x}_0 - \bar{x}, \underbrace{\bar{u} - \bar{u}}_0)$$

$\nearrow x^+ = Ax + Bu$

Since ϕ is the transition map of (1), the error satisfies (1) for zero input. This is (2).

Proof (alternative)

Since $\bar{x}$ verifies $\bar{x}^+ = \bar{x} = A\bar{x} + B\bar{u}$, compute $e^+ = x^+ - \bar{x}^+$ explicitly and obtain (2).

Remarks

- Stability/AS of $\bar{x}$ is the **same as** stability/AS of $\bar{e} = 0$ for (2).

Check: stability of $\bar{e} = 0$ means

$$\forall \varepsilon > 0, \exists \delta : \|e_0 - 0\| < \delta \implies \|e(k) - 0\| < \varepsilon, \forall k \geq 0$$

which, by using $e(k) = x(k) - \bar{x}$ coincides with the definition of stability for $\bar{x}$.

- (2) is independent of $\bar{u}$ and $\bar{x}$. This proves the following theorem.

Theorem

*An equilibrium state of an LTI system is stable/AS/unstable **if and only if** all other equilibria have the same properties.*

This is why we can say, for an LTI system, that « the system is stable ».

Stability and free states

For stability analysis, setting $u(k) = 0$ in $x^+ = Ax + Bu$ is not conservative. $\implies$ stability depends only on free states.

Theorem

An LTI system

- ① is stable $\iff$ all free states are **bounded**
- ② is AS $\iff$ all free states are **bounded and go to zero as $k \rightarrow +\infty$**
- ③ is unstable $\iff$ there is a free state which is **unbounded**

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Stability and eigenvalues of A

Each free state $x(k) = A^k x_0$ is a linear combination of the modes of A .

eigenvalues	modes
$\lambda_i \in \mathbb{R}$	$\begin{cases} 0 & \text{for } k < p_i \\ k^{p_i} \lambda_i^{k-p_i} & \text{for } k \geq p_i \end{cases}, \quad p_i = 0, 1, \dots, \eta_i - 1$
$\lambda_i = \rho_i e^{j\theta_i}$ AND $\lambda_h = \lambda_i^*$	$\begin{cases} 0 & \text{for } k < p_i \\ k^{p_i} \rho_i^{k-p_i} \sin(\theta_i(k - p_i) + \varphi_i) & \text{for } k \geq p_i \end{cases}$ $p_i = 0, 1, \dots, \eta_i - 1$

Stability and eigenvalues of A

Recall the macroscopic behaviour of modes

Lemma

- If $|\lambda_i| < 1$, all modes associated to λ_i are **bounded** and **go to zero** as $k \rightarrow +\infty$.
 - If $|\lambda_i| > 1$, all modes associated to λ_i are **unbounded**.
 - If $|\lambda_i| = 1$ and $\nu_i = n_i$, all modes associated to λ_i are **bounded**.
 - If $|\lambda_i| = 1$ and $\nu_i < n_i$, there's an **unbounded** mode associated to λ_i
-
- **Terminology:** « eigenvalues of A » = « eigenvalues of the system »
 - Combining the previous Lemma and Theorem we have the three Theorems given next

Stability and eigenvalues of A

Theorem (test of AS)

An LTI system is **AS if and only if** all its eigenvalues have modulus < 1 .

Theorem (test of instability)

An LTI system is **unstable if and only if** one of the following conditions occurs.

- 1 A system eigenvalue has modulus > 1 .
- 2 All system eigenvalues have modulus ≤ 1 and there is an eigenvalue λ_i with modulus $= 1$, algebraic multiplicity $n_i \geq 2$ and $\dim(V_{\lambda_i}) < n_i$.

Stability and eigenvalues of A

Theorem (test of simple stability)

An LTI system is **simply stable if and only if** all its eigenvalues have modulus ≤ 1 and, for each eigenvalue λ_i with modulus 1 and algebraic multiplicity $n_i \geq 2$, one has

$$\dim(V_{\lambda_i}) = n_i$$

- **Recall:** $\dim(V_{\lambda_i})$ is the geometric multiplicity of λ_i .
- **Remark:** AS is the most important property in engineering applications
- **Terminology:** we say that « A is Schur » if all its eigenvalues have modulus < 1 .

Exponential Stability

Recall that the equilibrium $\bar{x}$ of $x^+ = Ax$ is **exponentially stable** if there are $\alpha > 0$, $\rho \in [0, 1)$ such that

$$\|\tilde{x}(k) - \bar{x}\| \leq \alpha \rho^k \|\tilde{x}_0 - \bar{x}\|, \quad \forall \tilde{x}_0 \in \mathbb{R}^n$$

Lemma

An LTI system is AS if and only if it is ES.

Sketch of the Proof

AS $\iff$ all modes of the system go to zero as $k \rightarrow \infty$. But if a mode goes to zero, it does so exponentially fast. This implies that $\exists \alpha, \rho$ verifying the definition of ES

Examples

Analyse the stability of $x^+ = Ax + Bu$.

For $x \in \mathbb{R}^2$

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow \begin{cases} \lambda_1 = 2 \\ \lambda_2 = 0 \end{cases} \Rightarrow \text{unstable}$$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow \begin{cases} \lambda_1 = 1 \\ \lambda_2 = 0 \end{cases} \Rightarrow \text{stable but not AS}$$

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \Rightarrow \lambda_1 = \lambda_2 = 1 \Rightarrow \text{alg. multiplicity } n_1 = 2$$

$$V_1 = \left\{ v : A \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \right\} = \left\{ v : \begin{array}{l} v_1 + v_2 = v_1 \\ v_2 = v_2 \end{array} \right\} = \left\{ \begin{bmatrix} \alpha \\ 0 \end{bmatrix}, \alpha \in \mathbb{R} \right\}$$
$$\Rightarrow \dim(V_1) = 1 < n_1, \text{ therefore the system is unstable}$$

Section 2

Reachability and controllability

Key properties of dynamical systems

Reachability

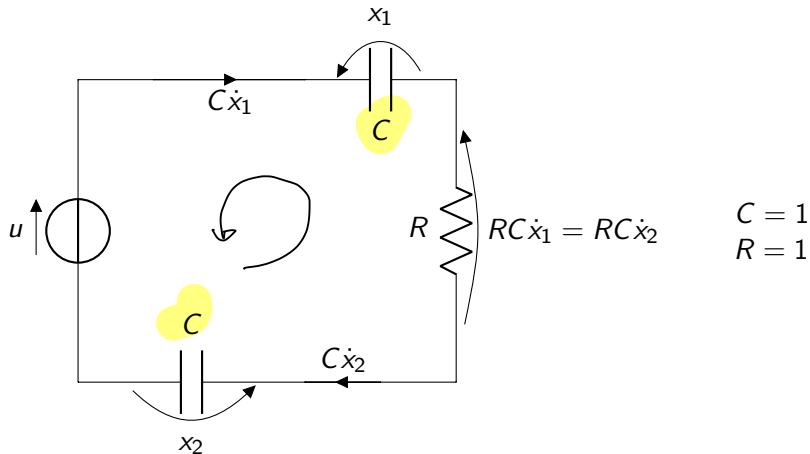
Is it possible to steer the state $x_0 = 0$ **to a desired value** by acting on the inputs?

Controllability

Is it possible to steer the state $x_0 \in \mathbb{R}^n$ **to the origin** by acting on the inputs?

Reachability

Example



Reachability

Example

Model

$$\begin{cases} u = x_1 + x_2 + \dot{x}_1 \\ u = x_1 + x_2 + \dot{x}_2 \\ y = x_1 \end{cases}$$

Discretization

$$\frac{dx}{dt} = \frac{x(k+1) - x(k)}{T}, \quad T = 0.1$$

$$\Rightarrow \begin{cases} x_1^+ = -9x_1 - 10x_2 + 10u \\ x_2^+ = -10x_1 - 9x_2 + 10u \\ y = x_1 \end{cases}$$

Can we modify x_1 independently of x_2 ? It seems not: in the CT model the currents in the upper and lower branches are identical.

Change of coordinates to highlight this phenomenon

$$\hat{x} = Tx, \quad T = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \Rightarrow \begin{cases} \hat{x}_1 = x_1 - x_2 \\ \hat{x}_2 = x_1 + x_2 \end{cases}$$

$$\hat{x}^+ = TAT^{-1}\hat{x} + TBu, \quad T^{-1} = \begin{bmatrix} +0.5 & 0.5 \\ -0.5 & 0.5 \end{bmatrix}$$

By direct calculation:

$$\begin{aligned} \hat{x}_1^+ &= \hat{x}_1 & \hat{x}_1(0) &= \hat{x}_1(1) = \dots \\ \hat{x}_2^+ &= -19\hat{x}_2 + 20u \end{aligned}$$

The difference of the voltages is constant and it cannot be affected by the input. $\Rightarrow$ states $\tilde{x} \in \mathbb{R}^2$ with $\tilde{x}_1 - \tilde{x}_2 \neq x_1(0) - x_2(0)$ cannot be reached.

Reachability: definitions

$$x^+ = Ax + Bu \quad (3)$$

$$y = Cx + Du \quad (4)$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$

Definition

A state $\tilde{x}$ is **reachable** if $\exists \tilde{k} > 0$ and $\tilde{u}(k)$, $k = 0, 1, \dots, \tilde{k}$ such that

$$x(\tilde{k}) = \phi(\tilde{k}, 0, 0, \tilde{u}) = \tilde{x} \quad (5)$$

If all states are reachable, then the system is termed « reachable ».

Reachability: definitions

Remarks

- Reachability = reachability from the origin as $x(0) = 0$ in (5)
- Reachability = property of the pair (A, B) only
- Problem: Difficult to check if a system is reachable using the definition (infinitely many $\tilde{x}$ should be tested)

Reachability test

Definition

The reachability matrix is defined as

$$M_r = [B \quad AB \quad A^2B \quad \dots \quad A^{n-1}B] \in \mathbb{R}^{n \times mn} \quad (6)$$

Remark: powers of A from 0 to $n - 1$ only.

Theorem

- ① $\tilde{x} \in \mathbb{R}^n$ is reachable only if it belongs to

$$X_r = \text{span}(M_r) \quad (7)$$

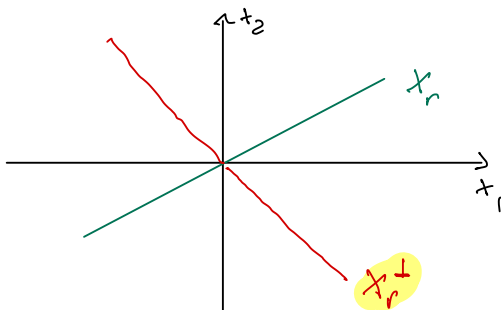
where $\text{span}(M_r)$ is the subspace spanned by the columns of M_r .

- ② If $\tilde{x}$ is reachable, it can be reached in $\tilde{k} \leq n$ steps.
- ③ The system is reachable if and only if $\text{rank}(M_r) = n$

Reachability test

Remarks

- For LTI systems, reachability is a finitely determined property.
- The orthogonal subspace $X_r^\perp$ is termed the « unreachable subspace ».
- The point 3 in the above theorem is a maximal rank condition



Reachability test

Proof

Setting $x(0) = 0$ one has

$$\begin{aligned} x^+ &= Ax + Bu \\ &\rightarrow \text{e.g. } [b_1 \ b_2] \begin{bmatrix} u_1(0) \\ u_2(0) \end{bmatrix} = u_1(0) b_1 + u_2(0) b_2 \\ x(1) &= Bu(0) \Rightarrow \text{lin. comb. of columns of } B \end{aligned}$$

$$\begin{aligned} x(2) &= Bu(1) + ABu(0) = [B \ AB] \begin{bmatrix} u(1) \\ u(0) \end{bmatrix} \\ &\Rightarrow \text{lin. comb. of columns of } [B \ AB] \end{aligned}$$

$\vdots$

$$x(k) = [B \ AB \ \dots \ A^{k-1}B] \begin{bmatrix} u(k-1) \\ u(k-2) \\ \vdots \\ u(0) \end{bmatrix}$$

Reachability test

Proof cont.

From the theorem of Cayley-Hamilton, if

$\psi(\lambda) = \lambda^n + \alpha_1 \lambda^{n-1} + \dots + \alpha_{n-1} \lambda + \alpha_n$ is the characteristic polynomial of A , then $\psi(A) = 0$, i.e.

$$A^n = -(\alpha_1 A^{n-1} + \alpha_2 A^{n-2} + \dots + \alpha_{n-1} A + \alpha_n I)$$

Therefore, the columns of $A^n B$ are a linear combination of the columns of matrices $A^i B$, $i = 0, 1, \dots, n-1$.

This shows that a state is reachable only if it can be reached in at most n steps and that the set of reachable states is given by (7).

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Unreachable systems

If the system is not reachable, the unreachable part can be isolated.

Theorem

Let $n_r = \text{rank}(M_r) \geq 1$. There is a suitable and non-unique change of state coordinates

$$\hat{x}(k) = T_r x(k), \quad \det(T_r) \neq 0$$

such that $x^+ = Ax + Bu$ is equivalent to

$$\hat{x}^+ = \hat{A}\hat{x} + \hat{B}u$$

(Handwritten red annotations: $T_r A T_r^{-1}$ above $\hat{A}$ and $T_r B$ below $\hat{B}$)

where

$$\hat{A} = \begin{bmatrix} \hat{A}_a & \hat{A}_{ab} \\ 0 & \hat{A}_b \end{bmatrix} \quad \hat{x} = \begin{bmatrix} \hat{x}_a \\ \hat{x}_b \end{bmatrix} \quad \hat{A}_a \in \mathbb{R}^{n_r \times n_r}$$

$$\hat{B} = \begin{bmatrix} \hat{B}_a \\ 0 \end{bmatrix} \quad \hat{B}_a \in \mathbb{R}^{n_r \times m}$$

$$\text{rank}([\hat{B}_a \quad \hat{A}_a \hat{B}_a \quad \dots \quad \hat{A}_a^{n_r-1} \hat{B}_a]) = n_r \quad (8)$$

Terminology: $(\hat{A}, \hat{B})$ is called the « reachability form » of (A, B)

Unreachable systems

The zero blocks in $(\hat{A}, \hat{B})$ reveal the **unreachable** part. Setting $\hat{x} = [\hat{x}_a^T \quad \hat{x}_b^T]^T$, $\hat{x}_a \in \mathbb{R}^{n_r}$, we have

$$\hat{x}_a^+ = \hat{A}_a \hat{x}_a + \hat{A}_{ab} \hat{x}_b + \hat{B}_a u \quad (9)$$

$$\hat{x}_b^+ = \hat{A}_b \hat{x}_b \quad (10)$$

- (9) is the reachable part. Since $(\hat{A}_a, \hat{B}_a)$ is reachable, one can steer $\hat{x}_a$ to an arbitrary position; (c.f. (8)).
- (10) is the unreachable part: $\hat{x}_b$ is not affected by u , neither directly, nor through $\hat{x}_a$.
- Terminology: the eigenvalues of $\hat{A}_a$ are termed « reachable ». Those of $\hat{A}_b$, « unreachable ». The same for the corresponding modes.

How to build T_r ?

- Build $M_r = [B \ AB \ \dots \ A^{n-1}B]$, $\text{rank}[M_r] = n_r$. Let $v_1, v_2, \dots, v_{n_r}$ be linearly independent columns of M_r .
- Build $T_r^{-1} = [v_1 \ \dots \ v_{n_r} \mid z_1 \ \dots \ z_{n-n_r}]$, where z_i are arbitrary vectors guaranteeing that $\det(T_r^{-1}) \neq 0$.

Example (ctd)

$$x^+ = \begin{bmatrix} -9 & -10 \\ -10 & -9 \end{bmatrix} x + \begin{bmatrix} 10 \\ 10 \end{bmatrix} u$$

$$M_r = [B \ AB] = \begin{bmatrix} 10 & -190 \\ 10 & -190 \end{bmatrix}, \quad n_r = 1$$

$$T_r^{-1} = \left[\begin{array}{c|c} 10 & -10 \\ 10 & 10 \end{array} \right] \implies T_r = \frac{1}{20} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$\hat{A} = T_r A T_r^{-1} = \begin{bmatrix} -19 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \hat{B} = T_r B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$\hat{x} = T_r x$
 $x = T_r^{-1} \hat{x} = [v_1 \dots v_{n_r} \mid *] \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix}$

Section 3

Controllability

Controllability

$$x^+ = Ax + Bu \quad (11)$$

$$y = Cx + Du \quad (12)$$

Definition

A state $\hat{x}$ is **controllable** if $\exists \hat{k} > 0$ and $\hat{u}(k)$, $k = 0, 1, \dots, \hat{k}$ such that

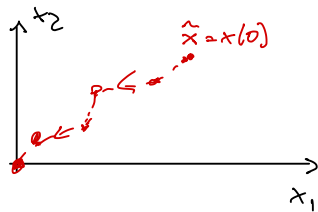
$$0 = \phi(\hat{k}, 0, \hat{x}, \hat{u})$$

If all states are controllable, then the system is termed «controllable».

Remarks

- Controllability = controllability to the origin
- Property of (A, B) only

Controllability



Controllability and reachability: do they coincide?

Example of a system which is controllable but not reachable

$$x^+ = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u, \quad \begin{array}{l} x(k) \in \mathbb{R}^2 \\ u(k) \in \mathbb{R} \end{array}$$

- Every state $\hat{x}$ is controllable using $u(\cdot) = \hat{u}(\cdot) = 0$

$$x(0) = \hat{x} = \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} \implies x(1) = \begin{bmatrix} \hat{x}_2 \\ 0 \end{bmatrix} \implies x(2) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- The state $\tilde{x} = \begin{bmatrix} 0 & 1 \end{bmatrix}^T$ cannot be reached by $x(0) = 0$.

$$x(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies x(1) = \begin{bmatrix} u(0) \\ 0 \end{bmatrix} \implies x(2) = \begin{bmatrix} u(1) \\ 0 \end{bmatrix} \implies \dots$$

- Intuition: all eigenvalues of A are zero $\implies$ free states go naturally to zero.

Controllability and reachability: do they coincide?

Lemma

One has:

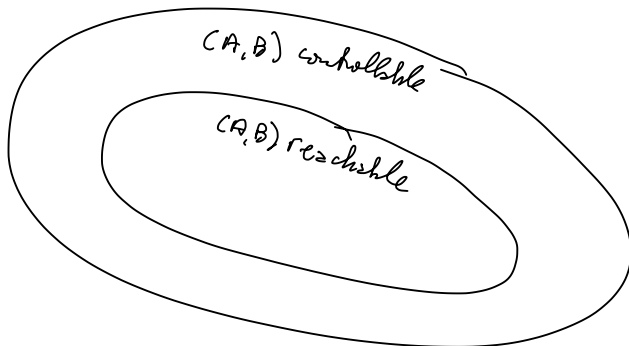
- i) (A, B) reachable $\implies (A, B)$ controllable
- ii) if $\det(A) \neq 0$:

$$(A, B) \text{ controllable} \implies (A, B) \text{ reachable}$$

Remark

LTI systems with $\det(A) \neq 0$ are termed **reversible**.

Controllability and reachability: do they coincide?



Controllability and reachability: do they coincide?

Proof of (i) (A,B) reachable $\Rightarrow (A,B)$ controllable

For $x(0) = \hat{x}$, setting $u^k = [u^T(k-1) \ \cdots \ u^T(0)]^T$, one has

$$x(k) = A^k \hat{x} + \underbrace{[B \ AB \ \cdots \ A^{k-1}B]}_{M_r^k} u^k$$

By setting $x(k) = 0$ one has

$$\hat{x} \text{ is controllable} \Leftrightarrow \exists u^k \text{ such that } -A^k \hat{x} = [B \ AB \ \cdots \ A^{k-1}B] u^k \quad (13)$$

Formula (13) is also the same as requiring that $-A^k \hat{x}$ is reachable from the origin in k steps. Equivalently,

$$-A^k \hat{x} \in \underbrace{\text{span}(M_r^k)}_{X_r^k} \quad (14)$$

If (A,B) is reachable, then $X_r^n = \mathbb{R}^n$ and (14) is verified for all $\hat{x} \in \mathbb{R}^n$ if $k = n$. This proves point (i).

Controllability and reachability: do they coincide?

Proof of (ii) If A is invertible, then (A,B) controllable $\Rightarrow (A,B)$ reachable

As for point (ii), by assumption, for any $\hat{x} \in \mathbb{R}^n$, $\exists k, \mathbf{u}^k$ s.t.

$$-A^k \hat{x} = M_r \mathbf{u}^k$$

Therefore, any $\bar{x}$ that can be written as

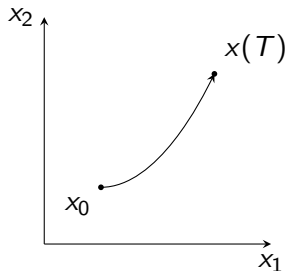
$$\bar{x} = -A^k \hat{x} \quad \text{for some } \hat{x}$$

is reachable. But since $\det(A) \neq 0$, the previous equation always has a solution $\hat{x}$ for any given $\bar{x} \in \mathbb{R}^n$.

Final remarks

A continuous-time LTI system is always reversible, meaning that $\det(e^{At}) \neq 0$ for any $A \in \mathbb{R}^{n \times n}$.

$$x(\tau) = e^{A\tau} x(0)$$



If $\exists u(t) \in [0, T]$ transferring x_0 into $x(T)$, then there is $u'(t) \in [0, T]$ transferring $x(T)$ into x_0 .

- Controllability and reachability coincide for continuous-time LTI systems
- Discrete-time LTI systems are substantially different

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