Multivariable Control (ME-422) - Exercise session 9

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1. Consider the system

$$\begin{cases} x_1^+ = (1-\alpha)x_1 + \beta x_2 - u + \tilde{d} \\ x_2^+ = \alpha x_1 + (1-\beta)x_2 \end{cases}$$
$$y = x_1$$

where $\alpha = 0.5$ and $\beta = 0.5$. Assume that the disturbance is generated by the LTI system

$$x_d^+ = 2x_d$$
$$\tilde{d} = x_d$$

Design a controller based on disturbance estimation for guaranteeing that $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \to 0$.

2. **Optimal Feedback Control of a Scalar System**
The plant to be controlled is the time-invariant scalar system

$$x_{k+1} = ax_k + bu_k \tag{1}$$

with performance index

$$J = \frac{1}{2}Sx_N^2 + \frac{1}{2}\sum_{k=0}^{N-1}\left(qx_k^2 + ru_k^2\right).$$

(a) Verify that the closed-loop system obtained by using the FH-LQ controller is given by

$$x_{k+1} = \frac{a}{1+\left(\frac{b^2}{r}\right)P_{k+1}}x_k$$
$$P_N = S$$
$$P_k = \frac{a^2 r P_{k+1}}{b^2 P_{k+1} + r} + q \tag{2}$$

Derive also the expression of the time-varying gain K_k and of the optimal performance index.

(b) Let $r = 0$, meaning that we do not care how much control is used (i.e., u_k is not weighted in J, so that the optimal solution will make no attempt to keep it small). Find the optimal control law, the optimal cost and relate the intuitive meaning of the cost to the behavior of the closed-loop system.

(c) If we are very concerned not to use too much control energy, we can let $r \to \infty$. In this case, find the closed-form expression of P_k as a function of S.
Hint: For $r \to +\infty$, (2) is an LTI system, for which one can apply the Lagrange formula.

Compute also the gain K_k and the optimal control law. Relate the intuitive meaning of the cost to the closed-loop dynamics.

(d) Write a MATLAB function

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[K, P] = fhopt(a,b,q,r,S,N)
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that computes and stores the values of optimal control gain K_k and optimal performance index P_k for $k = 0, \dots, N-1$.

Simulate the closed-loop system for $a = 1.05$, $b = 0.01$, $q = r = 1$, $x_0 = 10$, $S = 5$, $N = 100$ and plot the sequences P_k, K_k, and x_k. Is the state converging to zero? Why?

Set $S = 500$ and repeat the simulation. What is the difference?