Multivariable Control (ME-422) - Exercise session 5

Prof. G. Ferrari Trecate

1. Consider the LTI CT system

$$\begin{aligned}\dot{x} &= Ax + Bu \\ y &= Cx + Du\end{aligned}$$

As seen in the lectures, through Backward Euler (BE) discretization with sampling period $T > 0$ one obtains the DT system

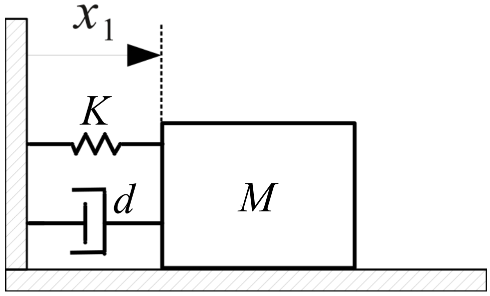
$$x_{k+1} = ATx_{k+1} + x_k + BTu_{k+1} \tag{1}$$
$$y_k = Cx_k + Du_k \tag{2}$$

(a) Show that, defining as a new state $w_k = x_k - TAx_k - TBu_k$, (1)-(2) become

$$\begin{aligned}w_{k+1} &= (I - AT)^{-1}w_k + (I - AT)^{-1}BTu_k \\ y_k &= C(I - AT)^{-1}w_k + \left[D + C(I - AT)^{-1}BT\right]u_k\end{aligned}$$

where we assumed that $I - AT$ is invertible.

(b) Consider the mass-spring-damper system



represented by the CT model

$$\begin{bmatrix}\dot{x}_1 \\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1 \\ -\frac{K}{M} & -\frac{d}{M}\end{bmatrix}\begin{bmatrix}x_1 \\ x_2\end{bmatrix} + \begin{bmatrix}0 \\ \frac{1}{M}\end{bmatrix}u$$

where $M = 1$, $K \geq 0$, and $d \geq 0$.

In the Exercise session 3, we have analyzed the DT model

$$\begin{bmatrix}x_1^+ \\ x_2^+\end{bmatrix} = \begin{bmatrix}1 & T \\ -\frac{K}{M}T & 1 - \frac{d}{M}T\end{bmatrix}\begin{bmatrix}x_1 \\ x_2\end{bmatrix} + \begin{bmatrix}0 \\ \frac{T}{M}\end{bmatrix}u,$$

produced by Forward Euler (FE) discretization with $T > 0$.

Compute the DT model obtained through BE discretization and analyze the stability properties in the following cases

- $T = 1$, $K = 0$, $d = 2$

- $T = 1$, $K = 0.5$, $d = 0$

Are the results different from those obtained for FE discretization? Which discretized model better captures the physical properties of the system?

2. Assume that the DT LTI system

$$x^+ = Ax + Bu \qquad x \in \mathbb{R}^n, u \in \mathbb{R}^m$$

is reachable and let $M_t = \begin{bmatrix} B & AB & \dots & A^{t-1}B \end{bmatrix}$ be the t-step rechability matrix.
For a given $\bar{x} \in \mathbb{R}^n$, among all inputs $u(k)$, $k = 0, \dots, t-1$ that steer $x(0) = 0$ to $x(t) = \bar{x}$, the one that minimizes

$$\sum_{k=0}^{t-1} \|u(k)\|^2$$

is given by

$$\begin{bmatrix} u_{ln}(t-1) \\ \vdots \\ u_{ln}(0) \end{bmatrix} = M_t^T \left(M_t M_t^T\right)^{-1} \bar{x} \tag{3}$$

The input u_{ln} is called *least-norm* (or *minimum-energy*) input.

(a) Show that $u_{ln}(k) = B^T (A^T)^{t-1-k} \left(\sum_{s=0}^{t-1} A^s BB^T \left(A^T\right)^s \right)^{-1} \bar{x}$ for $k = 0, \dots, t-1$.
Hint: Use the block structure of M_t for calculating the products in (3).

(b) Show that the minimum energy $\mathcal{E}_{min}(\bar{x}, t) = \sum_{k=0}^{t-1} \|u_{ln}\|^2$ required to reach $x(t) = \bar{x}$ is

$$\mathcal{E}_{min}(\bar{x}, t) = \bar{x}^T G_t^{-1} \bar{x}$$
$$G_t = \sum_{k=0}^{t-1} A^k BB^T \left(A^T\right)^k$$

The matrix G_t is called "t-steps reachability Gramian".

(c) $\mathcal{E}_{min}(\bar{x}, t)$ measures how hard it is to reach $x(t) = \bar{x}$ from $x(0) = 0$. In particular, the ellipsoid

$$\{\bar{x} \in \mathbb{R}^n : \mathcal{E}_{min}(\bar{x}, t) \leq 1\}$$

shows states reachable (in t steps) with one unit of energy. It also shows directions that can be reached with small inputs and directions that can be reached only with large inputs. Plot the ellipsoid in the state space for $t = 3$ and $t = 10$ when

$$A = \begin{bmatrix} 1.75 & 0.8 \\ -0.95 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{4}$$

(d) Using the data (4) plot $\mathcal{E}_{min}(\bar{x}, t)$as a function of t, with $\bar{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

(e) For general pairs (A, B) and target states $\bar{x}$, one has

$$t \geq s \implies \mathcal{E}_{min}(\bar{x}, t) \leq \mathcal{E}_{min}(\bar{x}, s) \tag{5}$$

i.e., it takes less energy to reach $\bar{x}$ over bigger time horizons.
Show that relation (5) holds when $A \in \mathbb{R}$ and $B \in \mathbb{R}$.