Multivariable Control (ME-422) - Exercise session 3

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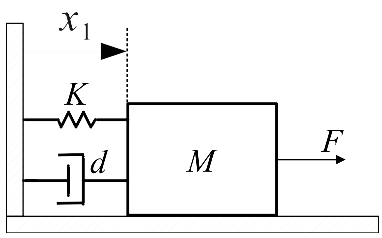
1. **Stability analysis**
   Consider the mass-spring-damper system



with $M > 0$, $K \geq 0$, and $d \geq 0$.

As seen in the lecture, a discrete-time model of the system is

$$\begin{bmatrix} x_1^+ \\ x_2^+ \end{bmatrix} = \begin{bmatrix} 1 & T \\ -\frac{K}{M}T & 1 - \frac{d}{M}T \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{T}{M} \end{bmatrix} u,$$

where $T > 0$ is the sampling time, $u = F$, and x_1 and x_2 are, respectively, the position and velocity of the mass. Set $T = 1$, $M = 1$, and, for each one of the following cases, analyze the stability of the system and compute the modes.

- $K = 0$, $d = 0$
- $K = 0$, $d \in (0, 1)$
- $K \in (0, 1)$, $d = 0$

In each case, check if the results match the physical intuition about the motion of the system.

2. **Stability analysis**
   Analyze by hand the stability of the system $x^+ = Ax$ where

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Hint: The eigenvalues of a block-diagonal matrix are the union of the eigenvalues of the blocks on the main diagonal.

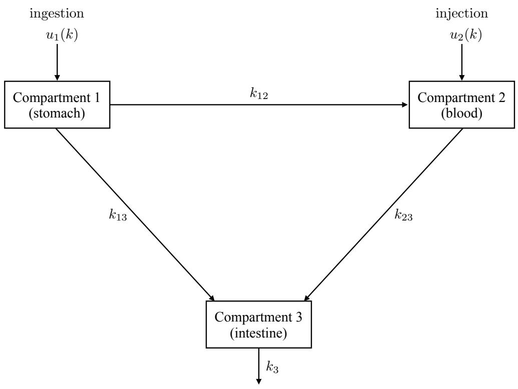
3. **Invariance of reachability**
   Consider the LTI system

$$x^+ = Ax + Bu \qquad x \in \mathbb{R}^n, \ \ u \in \mathbb{R}^m.$$

Show that the rank of the reachability matrix is invariant with respect to the change of coordinate $\hat{x} = Tx$, $\det(T) \neq 0$.

This means that the reachability properties of the equivalent LTI systems are the same.

4. Consider the multi-compartment system describing the discrete-time dynamics of a drug on various parts of the body, given in the figure, where the state $x_i(k)$ of each compartment is the mass of



the drug measured in mg in compartment i, and arrows represent mass transfer. The mass transfer rates are $k_{12} = k_{13} = k_{23} = k_3 = 0.5h^{-1}$. Inputs are mass flows measured in mg/h.

A discrete-time model describing the mass transfer between compartments is

$$
\begin{aligned}
x_1^+ &= -(k_{12} + k_{13})x_1 + x_1 + u_1 \\
x_2^+ &= k_{12}x_1 - k_{23}x_2 + x_2 + u_2 \\
x_3^+ &= k_{13}x_1 + k_{23}x_2 - k_3x_3 + x_3.
\end{aligned}
$$

(a)
- Remove the input u_1 from the model. Using MATLAB, check that the system is unreachable.
- Can you compute an unreachable state just by looking at the system structure? (**Hint:** Think about an experiment starting without drug in the body.)

Build the reachability form of the system and verify your result.

(b) Add back the input u_1 and check if the system is reachable, using MATLAB. If yes, compute a control law $u(k)$, $k = 0, 1, 2$ that, starting without drug in the body, produces $x_1(3) = 0.5$, $x_2(3) = 1$, and $x_3(3) = 1$.